SOLUTIONS OF THE EXAMPLES

IN

THE ELEMENTS

OF

STATICS AND DYNAMICS

BY

S. L. LONEY

CAMBRIDGE:
AT THE UNIVERSITY PRESS

Published July, 1893
Second Edition, 1899
Third Edition, with additions, 1902
Fourth Edition, with additions, 1906
Reprinted 1910, 1913, 1917, 1923, 1927, 1931. 1940

PRINTED IN GREAT BRITAIN

PREFACE.

For the following Solutions of the Examples in my *Elements of Statics and Dynamics* I am almost entirely indebted to a friend, to whom my best thanks are due He has also carefully revised the whole of the proof-sheets.

I hope these solutions will be found useful to teachers and private students.

S. L. LONEY.

Royal Holloway College,
Egham Surrey.
July 3, 1893.

The Fourth Edition has been altered so that the solutions correspond with the Tenth Edition of the *Elements of Statics and Dynamics.*

October 15, 1906.

ELEMENTS OF STATICS.—SOLUTIONS.

EXAMPLES. I. (Pages 15, 16.)

1. (i) $R = \sqrt{(24)^2 + 7^2} = \sqrt{625} = 25.$

(ii) $Q = \sqrt{(14)^2 - (13)^2} = \sqrt{27} = 3\sqrt{3}.$

(iii) $R = \sqrt{7^2 + 8^2 + 2 \,.\, 7 \,.\, 8 \cos 60^\circ} = \sqrt{169} = 13.$

(iv) $R = \sqrt{5^2 + 9^2 + 2 \,.\, 5 \,.\, 9 \cos 120^\circ} = \sqrt{61}.$

(v) $7^2 = 3^2 + 5^2 + 2 \,.\, 3 \,.\, 5 \cos a,$

whence $$\cos a = \frac{1}{2}, \text{ i.e. } a = 60^\circ.$$

(vi) $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \pm \frac{5}{13};$

$$\therefore R = \sqrt{(13)^2 + (14)^2 \pm 12 \,.\, 13 \,.\, 14 \,.\, \frac{5}{13}} = \sqrt{505}, \text{ or } 15.$$

(vii) $7^2 = 5^2 + Q^2 + 2 \,.\, 5 \,.\, Q \cos 60^\circ,$

whence $$Q^2 + 5Q - 24 = 0, \text{ and so } Q = 3.$$

2. The resultant of two forces is greatest or least according as they act in the same straight line in the same direction or in opposite directions. Hence [cf. Art. 23], the greatest resultant of forces of 12 lbs. wt. and 8 lbs. wt. $= (12 + 8)$ lbs. wt. $= 20$ lbs. wt.; and the least resultant $= (12 - 8)$ lbs. wt. $= 4$ lbs. wt.

3. The forces of 3 lbs. wt. and 4 lbs. wt. act in the same straight line in opposite directions, and are, therefore, equivalent to a force of 1 lb. wt. in the direction of the force of 4 lbs. wt., *i.e.* south. The forces of 5 lbs. wt. and 6 lbs. wt. also act in the same straight line in opposite directions, and are, therefore, equivalent to a force of 1 lb. wt. in the direction of the force of 6 lbs. wt., *i.e.* west. Hence the four forces are equivalent to two forces of 1 lb. wt. each, acting at right angles; therefore the resultant $= \sqrt{1^2 + 1^2} = \sqrt{2}$ lb. wt., and evidently acts along a line bisecting the angle between the lines of action of the forces of 4 lbs. wt. and 6 lbs. wt., *i.e.* in a direction south-west.

4. The resultant $= \sqrt{(84)^2+(187)^2} = \sqrt{42025} = 205$ lbs. wt.

5. Here $R=\sqrt{P^2+(P\sqrt{2})^2+2.P.P\sqrt{2}\cos 135^\circ}=P$ lbs. wt. at an angle $\tan^{-1}\dfrac{P\sqrt{2}\sin 135^\circ}{P+P\sqrt{2}\cos 135^\circ}$, *i.e.* $\tan^{-1}\infty$, *i.e.* at right angles to the direction of the first component.

6. If Q be the required force, we have

$$(2\sqrt{3})^2=2^2+Q^2+2.2.Q\cos 60^\circ,$$

whence $Q^2+2Q-8=0$, and therefore $Q=2$ lbs. wt.

7. If $\tan\alpha=\dfrac{12}{5}$, then $\cos\alpha=\dfrac{5}{13}$.

$$\therefore R=\sqrt{(13)^2+(11)^2+2.13.11.\frac{5}{13}}=\sqrt{400}=20 \text{ lbs. wt.}$$

8. If $\tan\alpha=\dfrac{4}{3}$, then $\cos\alpha=\dfrac{3}{5}$.

$$\therefore R=\sqrt{(10)^2+9^2+2.10.9\,\frac{3}{5}}=\sqrt{289}=17 \text{ lbs. wt.}$$

9. If the forces be each equal to P, α be the angle between them, and R be their resultant, we have

$$R^2=3P^2, \text{ so that } 3P^2=P^2(2+2\cos\alpha),$$

whence $\cos\alpha=\dfrac{1}{2}$, *i.e.* $\alpha=60^\circ$.

10. If P and Q be the required forces, we have

$$(\sqrt{10})^2=P^2+Q^2, \text{ and } (\sqrt{13})^2=P^2+Q^2+2PQ\cos 60^\circ,$$

i.e. $P^2+Q^2=10$, and $P^2+Q^2+PQ=13$.

Solving these equations, we have

$$P=3 \text{ lbs. wt., and } Q=1 \text{ lb. wt.}$$

11. (1) $P=P\sqrt{2(1+\cos\alpha)}$, *i.e.* $1=2+2\cos\alpha$;

whence $\cos\alpha=-\dfrac{1}{2}$, *i.e.* $\alpha=120^\circ$.

(2) $\dfrac{P}{2}=P\sqrt{2(1+\cos\alpha)}$, *i.e.* $\dfrac{1}{4}=2+2\cos\alpha$;

whence $\cos\alpha=-\dfrac{7}{8}$, *i.e.* $\alpha=\cos^{-1}\left(-\dfrac{7}{8}\right)=151^\circ\,3'$.

12. Here, if a be the required angle, we have

$$(\sqrt{A^2+B^2})^2=(A+B)^2+(A-B)^2+2(A+B)(A-B)\cos a,$$

so that $$A^2+B^2=2(A^2+B^2)+2(A^2-B^2)\cos a,$$

whence $$\cos a=-\frac{A^2+B^2}{2(A^2-B^2)}, \text{ i.e. } a=\cos^{-1}\left(-\frac{1}{2}\frac{A^2+B^2}{A^2-B^2}\right).$$

13. Find the resultant (R) of the two given forces; let S be the third given force; the greatest resultant R and S can have is $R+S$ when they act in the same direction in the same straight line; *i.e.* S must act in the direction of R.

14. Take the figure of Art. 27.

(i) Make $OA=5$ ins., $\angle AOB=37°$, and cut off $OB=7\frac{1}{2}$ ins.; complete the parallelogram $OACB$; then OC is R.

[2 units of force = one inch.]

(ii) Make $OA=4\frac{1}{2}$ ins., $\angle AOB=133°$ and cut off $OB=3\frac{1}{2}$ ins.; complete the parallelogram $OACB$; then $OC=R$.

(iii) Make $OA=3\frac{1}{2}$ ins.; with centres O and A describe circles of radii 5 and $2\frac{1}{2}$ ins. to meet in C; complete the parallelogram $OACB$; then $\angle AOB=a$.

(iv) Make $OA=3\cdot65$ ins. and $\angle AOB=65°$; draw AC parallel to OB; with centre O and radius $4\cdot35$ describe a circle to cut AC in C; complete the parallelogram $OACB$; then OB is Q.

EXAMPLES. II. (Pages 19, 20.)

1. The resolved parts are $10 \cos 30°$ and $10 \sin 30°$, respectively, *i.e.* $5\sqrt{3}$ lbs. wt. and 5 lbs. wt.

2. (1) $P \cos 45°$, *i.e.* $\frac{1}{2} P\sqrt{2}$. (2) $P \cos\left(\cos^{-1}\frac{12}{13}\right)$, *i.e.* $\frac{12}{13} P$.

3. The required force $=100 \cos 60° = 50$ lbs. wt.

4. If the required forces be each equal to P, we have

$$(100)^2 = P^2 (2 + 2 \cos 60°);$$

whence $3P^2 = (100)^2$, and $P = \frac{100\sqrt{3}}{3} = 57{\cdot}735$ lbs. wt.

5. If x and y be the required forces respectively, we have

$$\frac{x}{\sin 45°} = \frac{y}{\sin 60°} = \frac{50}{\sin 105°} = \frac{50}{\cos 15°} = \frac{50}{\cos (45° - 30°)}$$

$$= \frac{100\sqrt{2}}{\sqrt{3}+1} = 50\sqrt{2}\,(\sqrt{3} - 1).$$

Hence $$x = 50\,(\sqrt{3} - 1) = 36{\cdot}603 \text{ lbs. wt.},$$

and $$y = 25\,(\sqrt{18} - \sqrt{6}) = 44{\cdot}83 \text{ lbs. wt., nearly.}$$

6. If x and y be the required components respectively, we have

$$\frac{x}{\sin 45°} = \frac{y}{\sin 30°} = \frac{P}{\sin 75°} = \frac{P}{\sin (45° + 30°)} = \frac{P \times 2\sqrt{2}}{\sqrt{3}+1} = P\sqrt{2}\,(\sqrt{3} - 1).$$

Hence $$x = P\,(\sqrt{3} - 1), \text{ and } y = \frac{P}{2}\,(\sqrt{6} - \sqrt{2}).$$

7. The required force $= P\frac{\sin 45°}{\sin 60°} = P\sqrt{\frac{2}{3}}$.

8. If P and Q be the required forces respectively, we have

$$P = F \tan 60° = F\sqrt{3}, \text{ and } Q = F \sec 60° = 2F.$$

9. If a force F be resolved into two component forces P and Q, and P be at right angles to F and equal to it in magnitude, then the other angles are each 45°, and $Q=P\sqrt{2}=F\sqrt{2}$. Also the angle between the component forces is 135°.

10. Draw OB vertical and equal to 20 units of length, and OA horizontal and equal to 10 units of length. Complete the parallelogram $OABC$. Then OC represents the other force.

Clearly $OC=AB=\sqrt{(20)^2+(10)^2}=10\sqrt{5}=22\cdot36$ lbs. wt.

Also $\tan COB=\tan OBA=\frac{1}{2}$, so that the inclination to the vertical

$$=\tan^{-1}\frac{1}{2}=26°\,34'.$$

11. In Fig. Art. 34 make

$$OC=3\tfrac{1}{2} \text{ ins.}, \quad \angle COA=98° \text{ and } \angle COB=40°.$$

EXAMPLES. III. (Pages 25, 26.)

1. If P, Q and R be the forces, we have, by Lami's Theorem,

(i) $P=Q=R$.

(ii) $\dfrac{P}{\sin 150°}=\dfrac{Q}{\sin 150°}=\dfrac{R}{\sin 60°}$;

hence $$P:Q:R=1:1:\sqrt{3}.$$

2. If P, Q and R be the forces, we have, by Lami's Theorem,

$$\frac{P}{\sin 120°}=\frac{Q}{\sin 150°}=\frac{R}{\sin 90°};$$

hence $$P:Q:R=\sqrt{3}:1:2.$$

3. Since $7P$ is equal to the resultant of $5P$ and $8P$, we have, if a be the required angle,

$$(7P)^2=(5P)^2+(8P)^2+2\,.\,5P\,.\,8P\cos a,$$

whence $$\cos a=-\frac{1}{2}, \text{ i.e. } a=120°.$$

4. Draw a figure as in Art. 38, with $12P$ for P, $5P$ for Q, and $13P$ for R. The sides OL, LN and NO of the triangle OLN are proportional to 12, 5 and 13 respectively; and, since $(13)^2=(12)^2+5^2$, the angle OLN is therefore a right angle.

Again, $\tan LON=\frac{5}{12}=\cdot4166667$; therefore the angle $LON=22°\,37'$.

Hence the angle between the directions of the forces $5P$ and $12P$

$$=MOL=OLN=90°;$$

between the directions of the forces $12P$ and $13P$ the angle

$$=180°-22°\,37'=157°\,23',$$

and, therefore, between the directions of the forces $13P$ and $5P$ the angle $=112°\,37'$.

5. Construct a triangle ABC with its sides CA, AB, and BC proportional to 2, 3 and 4 respectively; and with the side BC in the given direction. The forces $2P$ and $3P$ are parallel to CA and AB.

6. The force represented by BE is the resultant of forces represented by BD and DE, *i.e.* by $\frac{1}{2}BA$ and $\frac{1}{2}BC$; the force represented by DC is the resultant of forces represented by $\frac{1}{2}AC$ and $\frac{1}{2}BC$; but, by the triangle of forces, the forces represented by $\frac{1}{2}BA$ and $\frac{1}{2}AC$ have resultant represented by $\frac{1}{2}BC$, therefore the required resultant is represented in magnitude and direction by $\frac{3}{2}BC$.

7. By the triangle of forces, the resultant is $\lambda . AB$, acting at P parallel to AB, *i.e.* is constant in magnitude and direction.

8. The diagonals of $ABCD$ bisect each other in some point O, and the resultant of the attractions to A and C is proportional to $2 . PO$, $= 2\lambda . PO$ suppose, and is in the direction PO; so the resultant of the repulsions from B and D is proportional to $2 . OP$ $(=2\lambda . PO)$ if the proportion be the same as for the attractions, and is in the direction OP. Hence P is in equilibrium independently of its position, *i.e.* wherever it is situated.

For Exs. 9- 14 take the figure of Page 13 with $\angle AOC = \theta$.

9. Make $OA = 5$ inches (scale 5 lbs. = one inch) and $\angle AOC = 35°$. With centre A and radius 4 inches describe a circle to cut OC in C_1, C_2. Complete the parallelograms OAC_1B_1 and OAC_2B_2. Then OC_1, OC_2 give the two values of R, and AOB_1, AOB_2 the two values of a.

10. Draw $OA = 5$ ins. [scale 10 kilog. = one inch]; with centres O and A and radii 7 and 6 ins. describe circles to meet in C. Complete the parallelogram $OACB$. Then AOB and AOC are the required angles a and θ.

11. Draw $OA = 3$ inches and $AOB = 130°$; with centre O and radius 4 ins. draw a circle to cut AC, parallel to OB, in C. Complete the parallelogram $OACB$; then $OB = Q$ and $\angle AOC = \theta$.

12. Draw $OA = 6$ ins., $AOB = 75°$ and $AOC = 40°$; through A draw AC parallel to OB; then AC is Q and OC is R.

13. Draw $OA=6$ ins., $\angle AOC=50°$ and make $OC=4$ inches. Join AC and complete the parallelogram $OACB$. Then OB is Q and AOB is a.

14. Draw $OA=4$ ins., $\angle AOB=55°$ and draw AC parallel to OB. With centre O and radius 5 ins. draw a circle to cut AC in C. Then $\angle AOC$ is θ and AC is Q.

15. Let OC be the direction of the boat's length; make $\angle AOC=20°$ and $OA=5$ ins. [Scale 1 cwt. = 1 inch.]

On the other side of OC from OA take OB such that

$$COB=180°-40°=140°.$$

Draw AC parallel to OB to meet OC in C; complete the parallelogram $OACB$; then on the given scale OC is the resultant force and OB the resultant reaction of the water.

EXAMPLES. IV. (Pages 26—28.)

1. Take the second figure in Art. 27 and we have $P=80$, $a=120°$, and $\angle COB=90°$. Let Q be the required force.

Since $\quad OB=BC\sin OCB=BC\sin COA=BC\sin 30°=\frac{1}{2}BC,$

therefore $$Q=\frac{1}{2}P=40.$$

2. Let $2P$ and P be the given forces, a be the angle between them, and R be their resultant. Then $R=2P$, and we have

$$(2P)^2=(2P)^2+P^2+2\,.\,2P\,.\,P\cos a,$$

whence $\quad \cos a=-\frac{1}{4}$, *i.e.* $a=\cos^{-1}\left(-\frac{1}{4}\right)$, *i.e.* $104°\ 29'$.

3. The resultant is always nearer to the greater force. Take the figure in Art. 38, with $P=3$ lbs. wt., the $\angle LOM=90°$, and the $\angle LOR=150°$; let Q and R be the required forces. Then the $\angle ROM=120°$, and we have

$$\frac{Q}{\sin 150°}=\frac{R}{\sin 90°}=\frac{3}{\sin 120°}, \text{ i.e. } 2Q=R=\frac{6}{\sqrt{3}},$$

$$\therefore\ Q=\sqrt{3}\text{ lb. wt., and } R=2\sqrt{3}\text{ lbs. wt.}$$

4. Take the second figure in Art. 27, with 30 lbs. wt. for P, $a=\frac{5}{3}\times 90°=150°$, and OC perpendicular to OB; Q and R being required. Then we have

$$R^2=(30)^2+Q^2+2.30.Q\cos 150°,$$

i.e. $$R^2=(30)^2+Q^2-30\sqrt{3}Q;$$

also, since $BC=OA$, and the angle BOC is a right angle, we have

$$R^2=(30)^2-Q^2.$$

$$\therefore\ (30)^2+Q^2-30\sqrt{3}Q=(30)^2-Q^2.$$

$$Q=15\sqrt{3}\text{ lbs. wt.}$$

Also, $$R^2=(30)^2-Q^2=(15)^2[2^2-(\sqrt{3})^2]=(15)^2,$$

so that $$R=15\text{ lbs. wt.}$$

Otherwise thus:—

$$OB=BC\cos OBC, \text{ i.e. } Q=30\cos 30°=15\sqrt{3}\text{ lbs. wt.}$$

Also, $$OC=BC\cos OCB, \text{ i.e. } R=30\cos 60°=15\text{ lbs. wt.}$$

5. Let $3P$ and $5P$ be the forces, and nP be their resultant. Take the second figure in Art. 27, with $3P$ for Q, nP for R, $5P$ for P, and OC at right angles to OB. Then, since $BC^2=OB^2+OC^2$, we have

$$(5P)^2=(3P)^2+(nP)^2,$$

$$\therefore\ 25=9+n^2, \text{ i.e. } n^2=16, \text{ and } n=4.$$

Hence $$5P : nP=5 : 4.$$

6. If P and Q be the forces, and R their resultant be perpendicular to Q, we have

$$P+Q=18 \quad (1) \qquad R=12\ .\ (2)$$

and $$P^2-Q^2=R^2, \text{ i.e. } (P+Q)(P-Q)=R^2 \ldots (3).$$

Substituting from (1) and (2) in (3), we have

$$18(P-Q)=144, \text{ i.e. } P-Q=8 \ldots (4).$$

From (1) and (4), we have $P=13$, and $Q=5$.

7. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$ with the diagonal OC (which represents R) equal to OA. Produce OA to D, making $AD=OA$, and complete the parallelogram $ODEB$; then OE represents the new resultant. Also $CE=CB=CO$; hence the angle BOE is a right angle, being an angle in a semicircle, and therefore OE is at right angles to OB.

8. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$; the diagonal OC represents the resultant $\sqrt{3}Q$, and the $\angle AOC=30°$. We have

$$\frac{\sin OAC}{OC}=\frac{\sin COA}{AC}, \quad i.e. \quad \frac{\sin OAC}{\sqrt{3}Q}=\frac{\sin 30°}{Q},$$

so that $\quad \sin OAC=\frac{\sqrt{3}}{2}$, *i.e.* the $\angle OAC=60°$ or $120°$.

If the $\angle OAC=60°$, then $OA=2\,.\,AC$, *i.e.* $P=2Q$.

If the $\angle OAC=120°$, then the $\angle ACO=30°$, *i.e.* $P=Q$.

9. Since the direction of the resultant is unaltered when the first force becomes $4P$ and the second force becomes $P+12$ lbs. wt., the ratios of the components in the two cases must be the same. Hence

$$\frac{2P}{P}=\frac{4P}{P+12}.$$

$$\therefore\ P+12=2P, \text{ and } P=12 \text{ lbs. wt.}$$

10. We have $(2m+1)^2(P^2+Q^2)=P^2+Q^2+2PQ\cos\theta$,

and $\quad (2m-1)^2(P^2+R^2)=P^2+Q^2+2PQ\sin\theta$;

$$\therefore\ (P^2+Q^2)(4m^2+4m)=2PQ\cos\theta,$$

and $\quad (P^2+Q^2)(4m^2-4m)=2PQ\sin\theta$;

$$\therefore\ \tan\theta=\frac{4m^2-4m}{4m^2+4m}=\frac{m-1}{m+1}.$$

11. Let α be the angle between P and Q; then we have

$$R^2=P^2+Q^2+2PQ\cos\alpha \quad \text{.............} \quad (1)$$

$$4R^2=P^2+4Q^2+4PQ\cos\alpha \quad \text{.........} \quad ..(2)$$

and $\quad 4R^2=P^2+Q^2+2PQ\cos(180°-\alpha)$,

i.e. $\quad 4R^2=P^2+Q^2-2PQ\cos\alpha \quad \text{....... ......} \quad ..(3).$

From (1) and (2), $\quad 2R^2=2Q^2-P^2$;

„ (2) „ (3), $\quad 12R^2=3P^2+6Q^2$, *i.e.* $4R^2=2Q^2+P^2$.

Hence, by addition, $\quad 6R^2=4Q^2$, *i.e.* $3R^2=2Q^2$;

and, by subtraction, $\quad 2R^2=2P^2$, *i.e.* $R^2=P^2$;

$$\therefore\ \frac{P^2}{2}=\frac{Q^2}{3}=\frac{R^2}{2}, \quad i.e.\ P:Q:R=\sqrt{2}:\sqrt{3}:\sqrt{2}.$$

12. Let the forces P and Q be represented by OA and OB respectively; complete the parallelogram $OACB$; the diagonal OC represents the resultant R. Produce OA to D, making $AD=OC$, and complete the parallelogram $ODEB$; join OE; then OE represents the resultant of $(P+R)$ and Q; and the $\angle COD=\theta$. Now $CE=AD=OC$; therefore the $\angle CEO=$the $\angle COE$. But the $\angle CEO=$the $\angle EOD$, since OD is parallel to CE; therefore the $\angle COE=$the $\angle EOD=\frac{\theta}{2}$. Q. E. D.

13. If AB represent the force P, which is turned through an angle α and then represented by AD, the system which was in equilibrium has had the force represented by AB taken away from it and the force represented by AD added to it. Hence, if BA be produced to C, so that $AC=AB$, the system is now equivalent to forces represented by AD and AC, which have a resultant (R, say) represented by AE, a diagonal of the parallelogram of which AC and AD are adjacent sides; and since $AD=AC$, therefore AE bisects the $\angle CAD$, so that the $\angle BAE=\frac{\pi}{2}+\frac{\alpha}{2}$. Hence, if α become $\alpha+\beta$, the $\angle BAE$ will become $\frac{\pi}{2}+\frac{\alpha+\beta}{2}$; *i.e.* if AD be turned through a further angle β, R turns through a further angle $\frac{\alpha+\beta}{2}+\frac{\alpha}{2}$ *i.e.* $\frac{\beta}{2}$, so that the inclination of R alters by half the amount that that of P does.

14. If OP be the line of action of the given force, and Q and R be the given points, the components will be equal if their directions are equally inclined to OP. Hence, if RM be perpendicular to OP and be produced to S so that $SM=MR$, then SQ or QS meets OP in T so that QT and RT are equally inclined to OP; and therefore if TP represent the given force in magnitude and UV be drawn through the centre of TP perpendicular to TP and meeting TR and TQ in U and V, the required components are TU and TV.

15. If A and B be the given points, and the forces P and Q along AC and BC meet in C, their resultant passes through C; and if the directions of the two forces be turned round A and B through equal angles CAD and CBD, the resultant will now pass through D, and meet its former direction in some point E. Also, since the $\angle CAD=$ the $\angle CBD$, a circle would go round $BACD$; and since the $\angle ACB=$ the $\angle ADB$, the resultant is unaltered in magnitude and must make the same angle ADE with AD as the angle ACE with AC, *i.e.* the $\angle ADE=$the $\angle ACE$; therefore a circle would go round $ACDE$. Hence, since there is but one circumscribing circle to the triangle ACD, E lies on the circle round ACB. Also the forces P and Q being given and the angle ACB always the same the direction of the resultant divides ACB into angles the ratio of whose series is known. Hence the angle BCE is known and therefore E is a fixed point.

16. By Art. 42, **Cor.**, the resultant of the forces represented by PA and PB must pass through F, the middle point of AB; also it passes through P and through C; hence P must lie on the straight line CF.

17. If P be the given force represented by AB acting at A, on a definite scale, then A and B are fixed; and if the other components be Q (which is invariable) represented by AC, and R, then, completing the parallelogram $ACBD$, R is represented by AD; also $BD=AC$, so that DB is constant; hence the locus of D is a definite circle with B as centre.

18. Let D, E and F be the middle points respectively of the sides BC, CA and AB of the triangle ABC, and P be any point. Then forces represented by PB and PC have a resultant represented by $2PD$; forces represented by PC and PA have a resultant represented by $2PE$; and forces represented by PA and PB have resultant represented by $2PF$. Hence the system $2PA$, $2PB$ and $2PC$ is equivalent to the system $2PD$, $2PE$ and $2PF$; *i.e.* the system PA, PB and PC is equivalent to the system PD, PE and PF.

19. Let $ABCD$ be the quadrilateral, and P be the required point. Join AC and BD, and bisect them in E and F, respectively. Then forces represented by PA and PC have resultant represented by $2PE$; and forces represented by PB and PD have resultant represented by $2PF$; hence, for equilibrium, PE and PF must lie in one straight line and be equal and opposite; therefore P is at the middle point of EF.

20. Through B draw a line parallel to AC to meet CD in L. Then forces represented by AB and BC are equivalent, by the triangle of forces, to a force represented by AC acting at B or at L.

The resultant of the first three forces is therefore the resultant of two forces acting at L represented respectively by AC and CD, *i.e.* by the triangle of forces, is represented by a force at L equal and parallel to AD.

Finally this force and the fourth force are equivalent to a force represented by $2AD$ acting at the middle point of DL.

21. By the polygon of forces, the force represented by AB is equivalent to forces represented by AH, HF and FB; the force represented by DC is equivalent to forces represented by DH, HF and FC; but the forces represented by AH and FB neutralise the forces represented by DH and FC, respectively; hence the resultant is parallel to HF and equal to $2HF$. No, see below.

22. The force represented by EG is equivalent to forces represented by EA, AD and DG; and the force represented by HF is equivalent to forces represented by HD, DC and CF. Also, the force represented by EG is equivalent to forces represented by EB, BC and

CG; and the force represented by HF is equivalent to forces represented by HA, AB and BF. Hence the system $2EG$ and $2HF$ is equivalent to the system AD and DC, and AB and BC, *i.e.* to AC and AC; and, therefore, the resultant of forces represented by EG and HF is represented by AC.

23. Through O the centre of the circle draw OC and OD perpendicular respectively to A_1PA_3 and A_2PA_4.

Then, since C is the middle point of A_1A_3, we have

$$PA_1 - PA_3 = PC + CA_1 - (CA_3 - PC) = 2PC.$$

Similarly $$PA_2 - PA_4 = 2PD.$$

The resultant of the four forces is therefore the resultant of $2PC$ and $2PD$, *i.e.* is represented by $4PE$, where CD meets PO in E.

Now, if α be the common inclination of A_1A_3 and A_2A_4 to the line PO, we have

$$PE = PC \cos\alpha = PO \cos^2\alpha.$$

Hence the resultant is independent of the radius of the circle.

EXAMPLES. V. (Pages 33—35.)

1. Here $$X = 1 + 2\cos 60° = 1 + 1 = 2,$$
$$Y = \sqrt{3} + 2\sin 60° = \sqrt{3} + \sqrt{3} = 2\sqrt{3};$$
$$\therefore\ F = \sqrt{X^2 + Y^2} = \sqrt{4 + 12} = 4 \text{ lbs. wt.},$$

and $$\tan\theta = \frac{2\sqrt{3}}{2} = \sqrt{3} = \tan 60°, \text{ i.e. } \theta = 60°,$$

so that the resultant is a force of 4 lbs. wt. in the direction AQ.

2. Taking the force of 5 lbs. wt. in the direction OX, the force of 3 lbs. wt. in the direction OY, and the force of 4 lbs. wt. in the direction bisecting the angle XOY, we have

$$X = 5 + 4\cos 45° = 5 + 2\sqrt{2},$$

and $$Y = 3 + 4\sin 45° = 3 + 2\sqrt{2};$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = \sqrt{50 + 32\sqrt{2}} = 9{\cdot}76 \text{ lbs. wt.},$$

and $$\tan\theta = \frac{3 + 2\sqrt{2}}{5 + 2\sqrt{2}} = \frac{7 + 4\sqrt{2}}{17},$$

$$\text{i.e. } \theta = \tan^{-1}\frac{7 + 4\sqrt{2}}{17} = 36° 40'.$$

3. Let the three forces be represented by the equal straight lines OB, OC and OA, so that the $\angle BOA$ = the $\angle COA = 60°$. The resultant of P in the direction OB and P in the direction $OC = 2P\cos 60° = P$, in the direction OA; therefore the required resultant is $2P$ in the direction OA.

4. Taking the force $13P$ in the direction OX, the force $10P$ in the second quadrant in the direction at 120° to OX, and the force $5P$ in the third quadrant in the direction at 120° to the directions of forces $13P$ and $10P$, we have

$$X = 13P - 10P\cos 60° - 5P\cos 60° = \frac{11}{2}P,$$

and $$Y = 10P\sin 60° - 5P\sin 60° = \frac{5\sqrt{3}}{2}P;$$

$$\therefore\ F = \sqrt{X^2+Y^2} = \frac{P}{2}\sqrt{121+75} = 7P;$$

also, if θ be the inclination of the direction of the resultant with the third force, $13P$, we have

$$\cos\theta = \frac{11}{2}P \div 7P = \frac{11}{14},\ i.e.\ \theta = \cos^{-1}\frac{11}{14} = 38° 13', \text{ nearly.}$$

5. Forces represented by $2P$, $2P$ and $2P$ in the given directions are in equilibrium [cf. the first figure in Art. 36], and may be removed, leaving forces represented by P and $2P$ acting at an angle of 120°. Hence, by Art. 27, we have

$$R = \sqrt{(2P)^2 + P^2 + 2\,.\,2P\,.\,P\cos 120°} = \sqrt{5P^2 - 2P^2} = P\sqrt{3}.$$

Again, since $(P\sqrt{3})^2 = (2P)^2 - P^2$, the $\angle ACB$ (Fig. Art. 36) is a right angle. Hence the $\angle CAB = 30°$, the $\angle CBA$ being 60°.

6. Through O draw the two fixed lines OX and OY perpendicular to BC and AB respectively. Let the force P_3 act along OY, and the force P_4 along OX, P_1 acting along OA in the second quadrant and P_2 along OB in the first quadrant. Let $P_1 = 4P$, $P_2 = 6P$, $P_3 = 5P$, and $P_4 = P$. Then we have

$$X = P_4 + P_2\cos 45° + 0 - P_1\cos 45°$$
$$= P\left(1 + \frac{6}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right) = P(1+\sqrt{2}),$$

and $$Y = 0 + P_2\sin 45° + P_3 + P_1\sin 45°$$
$$= P\left(\frac{6}{\sqrt{2}} + 5 + \frac{4}{\sqrt{2}}\right) = 5P(1+\sqrt{2});$$

$$\therefore\ F = \sqrt{X^2+Y^2} = P(1+\sqrt{2})\sqrt{26} = P \times 12{\cdot}31,$$

i.e. the resultant is proportional to 12·31, and if θ be the angle its direction makes with OX, *i.e.* with AB,

$$\tan\theta = \frac{Y}{X} = 5,$$

i.e. $$\theta = \tan^{-1}5,\ i.e.\ 78° 41'.$$

7. Here we have

$$X=1+6\cos 45^\circ=1+3\sqrt{2},$$
$$Y=9+6\sin 45^\circ=9+3\sqrt{2};$$
$$\therefore\ F=\sqrt{X^2+Y^2}=\sqrt{118+60\sqrt{2}}=14{\cdot}24 \text{ lbs. wt.}$$

8. Draw a figure similar to that on p. 31. Let the two forces of 4 lbs. wt. act in the directions OX and OB at 60° to each other, the force of 1 lb. wt. in the direction OC at 60° to OB, and the force of 3 lbs. wt. in the direction OX'. The force of 4 lbs. wt. in the direction OX and the force of 3 lbs. wt. in the direction OX' have resultant 1 lb. wt. in the direction OX; forces of 1 lb. wt. in the direction OX and 1 lb. wt. in the direction OC have resultant 1 lb. wt. in the direction OB [since they act at 120°, and, therefore, have resultant $=2\cos 60^\circ=1$]; therefore the resultant of the four given forces is 5 lbs. wt. in the direction OB. Hence, for equilibrium, the required force is 5 lbs. wt. in the direction opposite to OB, *i.e.* opposite to the direction of the second force.

9. If the angles between P and Q, Q and R, and R and $S=\alpha$, the angle between P and $S=3\alpha=108^\circ$, *i.e.* $\alpha=36^\circ$; and the forces being all equal, their resultant clearly acts in the direction bisecting the angle between Q and R. If each force be equal to P, the resultant of P and $S=2P\cos 54^\circ$; and the resultant of Q and $R=2P\cos 18^\circ$; hence the required resultant

$$=2P(\cos 54^\circ+\cos 18^\circ)=4P\cos 36^\circ\cos 18^\circ=\frac{P}{4}(\sqrt{5}+1)\sqrt{10+2\sqrt{5}}.$$

10. If $ABCDEF$ be the hexagon, and the given forces respectively act at A in the directions AB, AC, AD, AE and AF, the resultant obviously acts in the direction AD, and

$$=5+2\sqrt{3}\cos 30^\circ+2\,.\,2\cos 60^\circ=10 \text{ lbs. wt.}$$

11. Let $ABCDEF$ be the hexagon, and let the given forces respectively act at A in the directions AB, AC, AD, AE and AF. Take AB and AE coinciding with the fixed lines OX and OY. Then we have

$$X=2+3\cos 30^\circ+4\cos 60^\circ-6\cos 60^\circ=1+\frac{3\sqrt{3}}{2},$$

and $$Y=5+3\sin 30^\circ+4\sin 60^\circ+6\sin 60^\circ=\frac{13}{2}+5\sqrt{3};$$

$$\therefore\ F=\sqrt{X^2+Y^2}=\sqrt{125+68\sqrt{3}}=15{\cdot}58 \text{ lbs. wt.};$$

and $$\tan\theta=\frac{13+10\sqrt{3}}{2+3\sqrt{3}}=\frac{64+19\sqrt{3}}{23}=4{\cdot}213,$$

i.e. $$\theta=\tan^{-1}4{\cdot}213=76^\circ\,39'.$$

12. Let $ABCDE$ be the pentagon, and the forces of 7, 1, 1 and 3 lbs. wt. act along AB, AC, AD and AE respectively. Let XOX' and YOY' be the two fixed lines as on Page 31—OX coinciding with AB.

Then the $\angle ABC = \frac{1}{5}(5\pi - 2\pi) = \frac{3}{5}\pi$, *i.e.* the $\angle BAC = \frac{1}{2} \cdot \frac{2}{5}\pi = \frac{\pi}{5} = 36°$, the $\angle DAB = \frac{2}{5}\pi = 72°$, and the $\angle EAB = \frac{3}{5}\pi = 108°$.

Hence we have

$$X = 7 + 1 . \cos 36° + 1 . \cos 72° - 3 \cos 72°$$

$$= 7 + \frac{\sqrt{5}+1}{4} - 2 . \frac{\sqrt{5}-1}{4} = \frac{31-\sqrt{5}}{4},$$

$$Y = 1 . \sin 36° + 1 . \sin 72° + 3 . \sin 72°$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4} + 4 . \frac{\sqrt{10+2\sqrt{5}}}{4},$$

$$\therefore\ F = \sqrt{X^2 + Y^2} = \frac{1}{4}\sqrt{1136} = \sqrt{71} \text{ lbs. wt.}$$

13. If the equal forces P act on the angular point A of the octagon $ABCDEFGH$, in the directions AB, AC, AD, AE, AF, AG and AH, their resultant acts in the direction AE, by symmetry, and

$$= 2P \cos 67\tfrac{1}{2}° + 2P \cos 45° + 2P \cos 22\tfrac{1}{2}° + P$$

$$= P[\sqrt{2-\sqrt{2}} + \sqrt{2} + \sqrt{2+\sqrt{2}} + 1]$$

$$= P[\cdot 776 + 1\cdot 414 + 1\cdot 847 + 1]$$

$$= P \times 5\cdot 027.$$

14. Taking OX as the fixed line, we have

$$X = 11 \cos 18° 18' + 7 \cos 74° 50' - 8 \cos 49° 40'$$

$$= 10\cdot 4436805 + 1\cdot 8313939 - 5\cdot 1778672$$

$$= 7\cdot 0972072;$$

$$Y = 11 \sin 18° 18' + 7 \sin 74° 50' + 8 \sin 49° 40$$

$$= 3\cdot 4539175 + 6\cdot 7561823 + 6\cdot 0983352$$

$$= 16\cdot 3084350;$$

whence $F = \sqrt{X^2 + Y^2} = 17\cdot 79$ lbs. wt.,

and $\tan\theta = \frac{Y}{X}$,

so that $L \tan\theta = 10 + \log Y - \log X$, whence $\theta = 66° 29'$.

15.
$$X = 4\cos 20^\circ + 3\cos 40^\circ + 2\cos 60^\circ + 1\,.\cos 80^\circ$$
$$= 3{\cdot}75877 + 2{\cdot}29813 + 1{\cdot}00000 + {\cdot}17364$$
$$= 7{\cdot}23054;$$
$$Y = 4\sin 20^\circ + 3\sin 40^\circ + 2\sin 60^\circ + 1\,.\sin 80^\circ$$
$$= 1{\cdot}36808 + 1{\cdot}92836 + 1{\cdot}73205 + {\cdot}98481$$
$$= 6{\cdot}01330;$$
$$\therefore\ F = \sqrt{X^2+Y^2} = 9{\cdot}404 \text{ lbs. wt., and } \tan\theta = \frac{Y}{X} = {\cdot}8316,$$
whence $\theta = 39^\circ 45'$.

16.
$$X = 8\cos 30^\circ + 12\cos 70^\circ - 15\cos 59^\circ 45' - 20\cos 25^\circ$$
$$= 6{\cdot}92820 + 4{\cdot}10424 - 7{\cdot}55661 - 18{\cdot}12615$$
$$= -14\ 65032;$$
$$Y = 8\sin 30^\circ + 12\sin 70^\circ + 15\sin 59^\circ 45' + 20\sin 25^\circ$$
$$= 4 + 11{\cdot}27631 + 12{\cdot}95753 + 8{\cdot}45236$$
$$= 36{\cdot}68620;$$
$$\therefore\ F = \sqrt{X^2+Y^2} = 39{\cdot}506 \text{ lbs., wt ; also } \tan\theta = \frac{Y}{X} = -2{\cdot}5041.$$
$$\therefore\ \tan(180^\circ - \theta) = 2{\cdot}5041 = \tan 68^\circ 14', \text{ and } \theta = 111^\circ 46'.$$

17.
$$X = 85 + 47\cos 78^\circ - 63\cos 23^\circ$$
$$= 85 + 47 \times {\cdot}2079 - 63 \times {\cdot}9205 = 36{\cdot}78;$$
$$Y = 47\sin 78^\circ - 63\sin 23^\circ = 47 \times {\cdot}97815 - 63 \times {\cdot}3907$$
$$= 21{\cdot}359.$$
$$\therefore\ F = 42{\cdot}5 \text{ nearly, and } \tan\theta = \frac{21{\cdot}359}{36{\cdot}78} = {\cdot}5809, \text{ so that } \theta = 30^\circ 9'.$$

EXAMPLES. VI. Pages (38—41.)

1. If T_1 and T_2 be the required tensions respectively, we have, by Lami's Theorem,
$$\frac{T_1}{\sin 150^\circ} = \frac{T_2}{\sin 135^\circ} = \frac{W}{\sin 75^\circ} = \frac{W}{\sin(45^\circ + 30^\circ)}$$
$$= \frac{W.2\sqrt{2}}{\sqrt{3}+1} = W\sqrt{2}\,(\sqrt{3}-1),$$
whence $T_1 = \frac{W}{2}(\sqrt{6} - \sqrt{2})$, and $T_2 = W(\sqrt{3} - 1)$.

2. Let A be the fixed point, C the end of the string (tension T) to which the body is attached, so that $AC=25$ ins.; and let the line of action of the horizontal force F meet the vertical through A in B, so that $CB=20$ ins. Since ABC is a right angle,

$$AB=\sqrt{(25)^2-(20)^2}=15 \text{ ins.}$$

Then the triangle ABC is the triangle of forces, and

$$\frac{F}{BC}=\frac{T}{CA}=\frac{2}{AB},$$

i.e.

$$\frac{F}{20}=\frac{T}{25}=\frac{2}{15},$$

$$\therefore\ F=\frac{8}{3}=2\tfrac{2}{3} \text{ lbs. wt., and } T=\frac{10}{3}=3\tfrac{1}{3} \text{ lbs. wt.}$$

3. As in Ex. 1, p. 36, with $AC=16$ ins., $BC=63$ ins., $AB=65$ ins., and 130 lbs. attached at C, we have

$$T_1=126 \text{ lbs. wt., and } T_2=32 \text{ lbs. wt.}$$

4. Here, with $AC=6$ ft., $BC=8$ ft., $AB=10$ ft., and 70 lbs. attached at C, we have

$$T_1=56 \text{ lbs. wt., and } T_2=42 \text{ lbs. wt.}$$

5. Here, with $AC=9$ ft., $BC=12$ ft., $AB=15$ ft., and 60 lbs. attached at C, we have

$$T_1=48 \text{ lbs. wt., and } T_2=36 \text{ lbs. wt.}$$

6. The tension = the weight supported. Hence the tension in the lowest part = 4 lbs. wt.; in the middle part it = (4 + 4) lbs. wt. = 8 lbs. wt.; and in the highest part it = (4 + 4 + 4) lbs. wt. = 12 lbs. wt.

27 7. The tension of the string is equal to W throughout. Hence the pressure on each tack = the resultant of two equal forces W acting at an angle of $120° = 2W\cos 60° = W$.

8. Let A and B be the positions of the men, and C be the point of the boat to which the ropes are attached. Since the ropes are of the same length, A and B are opposite points on the two banks. The forces of 100 lbs. wt. act in the directions CA and CB, and the resultant pressure acts in the direction CD, where D is the middle point of AB. Now

$$AD=DB=48 \text{ feet, and } CA=CB=60 \text{ feet,}$$

$$\therefore\ CD=\sqrt{(60)^2-(48)^2}=36 \text{ ft.}$$

Hence the resultant pressure $=2\times 100\times\cos\left(\frac{1}{2}ACB\right)$

$$=2\times 100\times\frac{36}{60}=120 \text{ lbs. wt.}$$

9. Let the weights be each equal to W, and A be the point in the string to which the third weight is fastened. Also let B and C be the points of the two bars in contact with the string. A is under the action of three equal forces, the directions of which, for equilibrium, must make angles of 120° with one another; thus AB and AC are each inclined at an angle of 60° to the vertical. Hence the pressure on each bar = the resultant of two equal forces W acting at an angle of $60° = 2W\cos 30° = W\sqrt{3}$.

10. The tension T in the two portions of the string is the same. Resolving vertically and horizontally, we have

$$T\cos 75° + T\cos 45° = 27,$$

and
$$P + T\sin 45° = T\sin 75°,$$

$$\therefore\ T = \frac{27}{\cos 75° + \cos 45°} = \frac{27}{2\cos 60°\cos 15°} = \frac{27}{\cos 15°};$$

and
$$P = T(\sin 75° - \sin 45°) = 2T\cos 60°\sin 15° = T\sin 15°;$$

$$\therefore\ P = 27\tan 15° = 27(2-\sqrt{3}) = 7{\cdot}23 \text{ lbs. wt.}$$

11. If A be the highest point of the circle, O its centre, and if the rings be at B and C, then the angle $AOB = 30°$ = the angle AOC; also the tension T of the string, being the same throughout, must be equal to each of the outer weights W'. For the equilibrium of the ring B, the parts of the string must make equal angles with the normal BO; but the vertical part makes an angle of 30°; therefore, if the middle weight W be at D, the angle $DBO = 30°$ = the angle DCO, similarly. Thus BD and CD being drawn at equal angles to BO and CO, D must lie in the bisector of the angle BOC, *i.e.* in AO; also the angle $ADB = 60°$ = the angle ADC,

so that
$$W = 2T\cos 60° = T = W';$$

and, therefore, the three weights are equal.

12. Let DC be the required depth, D being the middle point of AB when the mass of 5 lbs. is attached at C. Then 5 lbs. wt. is the resultant of two forces each equal to 112 lbs. wt., and we have

$$5 = 2\,.\,112\,.\cos BCD = 224\,.\frac{CD}{CB};$$

hence, if $DC = x$,

$$\frac{5}{224} = \frac{x}{\sqrt{5^2+x^2}}, \text{ and } x = 1{\cdot}34 \text{ ins.}$$

If the small mass were attached to any other point of the string, the knot C would move obliquely downwards until the position above is reached, since equilibrium is only possible when CA and CB are equally inclined to the horizontal, and therefore C vertically below D.

13. Let AB be the rod, D be its middle point, and AC and BC be the strings, so that $AC = 7$ ins., $BC = 24$ ins., and $AB = 25$ ins. C, the point of suspension of the body, is vertically below D. Since

$$(25)^2 = 7^2 + (24)^2,$$

the angle ~~ABC~~ ACB is a right angle, and $DC = AD = DB$. Draw DE parallel to AC, bisecting BC in E, and let T_1 and T_2 be the tensions of the strings CA and CB respectively. Then the triangle CED is a triangle of forces for T_1, T_2 and 10 lbs. Hence

$$\frac{T_1}{DE} = \frac{T_2}{CE} = \frac{10}{CD}, \text{ i.e. } \frac{T_1}{\left(\frac{7}{2}\right)} = \frac{T_2}{12} = \frac{10}{\left(\frac{25}{2}\right)};$$

$$\therefore\ T_1 = 7 \times \frac{2}{5} = \frac{14}{5} = 2\tfrac{4}{5} \text{ lbs. wt.,}$$

and $$T_2 = 12 \times \frac{4}{5} = \frac{48}{5} = 9\tfrac{3}{5} \text{ lbs. wt.}$$

14. Let A and B be the ends to which the weights of 10 lbs. and 16 lbs. are attached respectively, and C be the highest point of the chain on the pulley. Then the weight of AC + 10 lbs. = 20 lbs. wt.,

i.e. the weight of AC = 10 lbs.;

also the weight of BC + 16 lbs. = 20 lbs. wt.,

i.e. the weight of BC = 4 lbs.;

$\therefore$ the weight of the chain = (10 + 4) lbs. = 14 lbs.

15. The whole weight hanging on the peg = (15 + 7) lbs. = 22 lbs. Hence the longer length of the hanging chain must weigh $\frac{22}{2}$, *i.e.* 11 lbs., and the shorter length 4 lbs.

$\therefore$ the longer length : the whole length = 11 : 15;

i.e. the longer length $= \frac{11}{15} \times 8\frac{3}{4} = \frac{77}{12}$ feet = 6 ft. 5 ins.;

and hence the shorter length = 2 ft. 4 ins.

16. If C be the centre, and A be the highest point of the circle; P be the body and W its weight; T be the tension of the string PA; and R be the reaction of the wire on P (this, being normal, acts along CP), then P is in equilibrium under the action of three forces through P, parallel to the sides of the triangle ACP, which is equilateral; therefore the three forces are equal, *i.e.* $T = R = W$.

17. The rhombus is a pair of equilateral triangles; also the resultant of P and Q is vertical, since it balances W the weight of the lamina. Resolving horizontally, we have

$$P \cos 60° = Q \cos 30°, \text{ i.e. } P \cdot \frac{1}{2} = Q \cdot \frac{\sqrt{3}}{2};$$

hence $$P^2 = 3Q^2.$$

18. Since the whole pull of the driver is equal to P, the tension in each side is equal to $\frac{P}{2}$, and this tension is the same throughout since the rings are smooth. The resultant of $\frac{P}{2}$ and $\frac{P}{2}$ acts along the line bisecting the angle a between the portions of the rein, and is equal to $2 \cdot \frac{P}{2} \cdot \cos\frac{a}{2}$, *i.e.* $P\cos\frac{a}{2}$. Thus the whole pull is equal to $2P\cos\frac{a}{2}$.

19. Let T be the tension of the string BC, T_1 be the tension of the string AB or the string AC (by symmetry the same), and T_2 be the tension of the supporting string at B or C. For the equilibrium of the whole, resolving vertically, we have

$$2T_2\cos 45^\circ = W, \text{ i.e. } T_2 = \frac{W}{\sqrt{2}}.$$

At A, resolving vertically, we have

$$2T_1\cos 30^\circ = W, \text{ i.e. } T_1 = \frac{W}{\sqrt{3}}.$$

At C, resolving horizontally, we have

$$T + T_1\cos 60^\circ = T_2\cos 45^\circ, \text{ i.e. } T + \frac{W}{2\sqrt{3}} = \frac{W}{2},$$

whence

$$T = \frac{W}{6}(3 - \sqrt{3}).$$

20. The forces at A and B are equal ($=P$, say). Since the direction of the force P bisects the angle ABC, the tension of the string BC is equal to the tension of the string AB ($=T$, say). Hence, for the equilibrium of C, resolving vertically, we have

$$2T\cos\frac{C}{2} = W, \text{ i.e. } T = \frac{W}{2}\sec\frac{C}{2}.$$

21. (1) The ring being smooth, T the tension of the string is the same throughout, and W the weight of the ring, acting vertically downwards, is the resultant of two equal forces T; hence the portions of the string are equally inclined to the vertical.

(2) If the ring be fixed, let W be its weight, acting vertically downwards, and T_1 and T_2 be the tensions of the portions of the strings inclined at angles θ and ϕ to the vertical respectively. Since the ring is kept in equilibrium by the three forces T_1, T_2 and W, we have, by Lami's Theorem,

$$\frac{T_1}{\sin\phi} = \frac{T_2}{\sin\theta} = \frac{W}{\sin(\theta+\phi)};$$

whence

$$\frac{T_1}{T_2} = \frac{\sin\phi}{\sin\theta}.$$

22. Let $ABCDE$ be the loop, B and C be the positions of the two higher pegs, A and D be the positions of the two lower pegs, and E be the lowest point of the loop. Then

$$\angle ABC = \angle BCD = 120^\circ;$$

and
$$\angle BAE = \angle CDE = 90^\circ.$$

Also, if EF be drawn vertically upwards to meet BC in F, the $\angle BFE = 90^\circ$. Hence a circle would go round $BAEF$; and the $\angle AEF = 180^\circ - \angle ABF = 60^\circ$, and therefore the $\angle AED = 120^\circ$. Therefore, if T be the tension of the string, which is the same throughout, resolving vertically for W, we have

$$W = 2T\cos 60^\circ, \text{ i.e. } T = W.$$

Also, the pressure on B or C, due to two equal forces T acting at an angle of 120°

$$= 2T\cos 60^\circ = T = W;$$

and the pressure on A or $D = 2T\cos 45^\circ = W\sqrt{2}$.

23. Let O be the fixed point in the middle of the stream, C the centre of the boat, P the resultant pressure of the current on the boat and θ the angle ACO, the direction of the current being AC.

The component $P\cos\theta$ along the line OC is neutralised by the tension of the rope. The perpendicular component $P\sin\theta$ urges the boat at right angles to OC and causes it to describe an arc of a circle round O as centre.

When the boat has gone past the middle of the stream the effect of the current is to stop the motion of the boat, as may be seen by drawing a figure.

This is not a practical contrivance for crossing a river unless the stream be a fast-running stream, such as the Rhine and other continental rivers.

EXAMPLES. VII. (Pages 45, 46.)

1. Scale—20 feet = 1 inch, and 20 lbs. = 1 inch. Let A and C be the positions of the men, and B be the point of the boat to which the ropes are attached, so that AB represents 30 ft., AC represents 50 ft., and CB represents 45 ft. B is found by describing circles round A and C, of radii proportional to 30 ft. and 45 ft. respectively, on scale. Produce AB to D, so that BD represents 5 ft. and therefore AD represents the tension 35 lbs. wt. on the assumed scale; draw DE perpendicular to AC, making AE parallel to BC; then DE represents the force P of the stream, and AE represents the tension T of the second rope; also AD represents 35 lbs., and the lengths of DE and AE in inches being measured (by a Diagonal Scale), their lengths multiplied by 20 give the forces in lbs. wt.

2. Scale—4 feet $=1$ inch. Let B be the foot of jib; draw BC vertically upwards to represent 6 feet, and with centre B and radius 10 feet, on scale, describe a circle cutting the perpendicular through C to BC in A; then AB represents the jib, and AC the tie-rod. If now T be the tension of the tie-rod, and T' be the thrust on the jib, the mass of 1 ton is supported by T acting towards C, and T' acting towards A; and thus we have ABC a triangle of forces, CB representing 1 ton. Thus

$$T=1\times1\tfrac{1}{3}=1\tfrac{1}{3}\text{ ton wt.},$$

and $$T'=1\times1\tfrac{2}{3}=1\tfrac{2}{3}\text{ ton wt.}$$

3. Scale—2 feet$=1$ inch. Take A the upper of the two fixed points; draw AD, horizontal, $=2$ ins., and BD, perpendicular to AD, $=\dfrac{1}{2}$ in. Circles round A and B with radii $2\frac{1}{2}$ ins. and $1\frac{1}{2}$ ins. respectively meet in C. Draw CE, downwards, parallel to DB, $=2$ ins., and EF parallel to CB, meeting AC (produced) in F; the tensions T and T' of the strings are represented by CF and FE on the same scale as CE represents 1 cwt., *i.e.* on the scale 112 lbs. $=2$ ins., *i.e.* 7 lbs. $=\dfrac{1}{8}$ in. Thus the lengths of CF and FE in eighths of an inch, multiplied by 7, give T and T' in lbs. wt.

4. Scale—4 feet$=1$ inch. Draw $AD=3\frac{1}{2}$ ins.; AE and DF, vertically downwards, equal to 1 in. and $1\frac{1}{2}$ ins. respectively; and EB and FC parallel to AD, meeting the circles round A and D with radii $1\frac{1}{2}$ ins. and 2 ins. in B and C, respectively. Draw CG, vertically downwards, equal to 1 in.; then, on the scale 4 lbs.$=1$ in., CG represents the weight at B; and if GH be drawn parallel to BA, meeting BC (produced) in H, then HC represents the tension T in BC on the same scale; also, if HK be drawn parallel to CG, meeting DC (produced) in K, then the weight at C is represented by HK on the scale of 4 lbs. to 1 inch.

5. Draw the triangle ABC of the dimensions given.

Draw KL vertically downwards and equal to 4 inches, to represent 10 cwt. on the scale $2\frac{1}{2}$ cwt.$=1$ inch. Draw LM, KM parallel to AC, CB to meet in M; then LM, MK represent the thrusts T_1 and T_2 in AC and BC. Draw MN horizontal to meet KL in N. Then KMN is the triangle of forces for the joint B, so that MN represents the action T_3 in BA, and NK represents the action S at B; hence R, the vertical action at A is given by LN. AB is a tie; AC, CB are struts.

6. The figure ABC being described, draw a vertical line KL 4 inches long to represent 200 lbs. (scale 50 lbs.$=1$ inch); then draw LM, KM parallel respectively to BC, AC; the triangle LMK is now a triangle of forces for the point C. On measurement $LM=2{\cdot}4$ and $KM=3{\cdot}2$ inches, so that the respective thrusts in BC, AC are $50\times2{\cdot}4$ and $50\times3{\cdot}2$, *i.e.* 120 and 160 lbs. Through M draw NMO a vertical line, and horizontal lines KN, LO. Then KMN, and MLO

are triangles of forces for the points A, B respectively, so that MN, OM represent the actions at A and B. We find $MN=2{\cdot}54$ and $OM=1{\cdot}44$ ins. so that the actions are 128 and 72 lbs. wt.

7. The triangle formed by the post, jib, and tie is a triangle of forces, so that

$$\frac{\text{Thrust in jib}}{20}=\frac{\text{Pull in the tie}+10\text{ cwts.}}{16}=\frac{10\text{ cwts.}}{10}.$$

Hence the result.

8. The triangle ACD is a triangle of forces, so that

$$\frac{\text{Thrust along } AC}{11}=\frac{500-\text{thrust along } DC}{5}=\frac{500}{AD}=\frac{500}{\sqrt{96}}\text{ etc.}$$

9. The triangle ACD is a triangle of forces. If T be the pull along CD, the total force in that direction is $T+1$, since the tension of the chain is everywhere one ton. On drawing KL vertical and equal to one inch to represent one ton, and then LM and KM parallel to AC and DC we have on measurement $KM=1{\cdot}93$ ins. and $LM=2{\cdot}73$ ins.

$$\therefore\ T+1=1{\cdot}93 \text{ tons and } R=2{\cdot}73 \text{ tons.}$$

10. The tension of the chain all through is one ton and ACD is a triangle of forces:

(1) Here $\dfrac{R-1}{80}=\dfrac{T}{20}=\dfrac{1}{15}$, giving R and T.

(2) Here $\dfrac{R_1}{80}=\dfrac{T_1+1}{20}=\dfrac{1}{15}$, giving R_1 and T_1.

11. On drawing the figure with the dimensions given, and then the right-hand figure of Ex. 3, Page 44 with KL, LM each two inches to represent two tons we have, on measurement, $MN=5{\cdot}01$ inches and $NK=1{\cdot}8$ inches. Hence the result.

12. 25 lbs.$=1$ inch. Draw CL vertical (in the direction of AC produced) and equal to 4 inches. Draw CM, LM parallel to AD and CD respectively. Then CML is a triangle of forces for the joint D and thus LM represents the tension in CD. Through L draw LN parallel to BC and make LN equal to LM. By symmetry, LN represents the tension in BC. Then NM will be vertical, and LNM a triangle of forces for the joint C, the length NM representing the tension in CA. On measurement, NM will be found to be $2{\cdot}08$ inches and thus the corresponding tension $25\times 2{\cdot}08$, that is, 52 lbs. wt.

13. 25 lbs. wt.$=1$ inch. Draw CE vertically downwards and equal to 4 inches. Draw EF parallel to CD and produce BC to meet it in F. Then EFC is a triangle of forces for the joint C, so that EF represents the action in the rod CD. Draw FG parallel to AD and EG horizontal and let them meet in G. Then EGF is a triangle of

forces for the joint D, so that EG represents the action in BD. On measurement $EG=3\cdot08$ inches nearly, so that the thrust required $=25\times3\cdot08=77$ lbs. wt. nearly.

14. 25 lbs. wt. $=1$ inch; produce DB to E making BE one inch. Draw EF parallel to CD to meet CB produced in F. Then EBF is a triangle of forces for the joint B, so that EF represents the action in the rod BA. Draw FG parallel to the rod DA making FE and FG equal; then by symmetry FG represents the action in the rod AD. Join GE. Then GE is parallel to AC and represents the action in AC. On measurement $GE=1\cdot26$ inches nearly. Thus tension of $AC=25\times1\cdot26$ lbs. wt. $=$ about $31\cdot6$ lbs. wt.

15. Draw CE vertical, 2 inches in length, to represent 100 lbs. Draw EM, MC parallel to CD, CB respectively. Then EM represents the action along CD and therefore ME represents the action of CD on DA. The other forces on AD are T in direction GF, where G and F are the middle points of AB, AD; X the horizontal reaction of AB on AD at A; and 50 lbs. wt., equal to one-half the supporting tension, at A.

Draw ML perpendicular to EC; then EL represents the 50 lbs. wt. just mentioned. Thus LM represents the resultant of X at A, and T at F; also, since the vertical line through A meets the reaction at D in C, the resultant of these two horizontal forces acts through C.

Hence, since C is four times as far from A as the line GF is, we have

$$X.AC=T.HC \text{ [where } AC \text{ meets } GF \text{ in } H\text{]},$$

$$i.e.\ 4X=3T. \quad \therefore\ T-X=\tfrac{1}{4}.T;$$

$\therefore$ LM represents $\frac{1}{4}T$.

But, on measurement, $LM=\cdot577$ inches.

$$\therefore\ T=4\times\frac{\cdot577}{2}\times100 \text{ lbs. wt.}=115\cdot4 \text{ lbs. wt.}$$

EXAMPLES. VIII. (Pages 55—57.)

1. (i) $$R=P+Q=11,$$

and $$4.AC=7.BC=7(11-AC), \text{ whence } AC=7 \text{ ins.}$$

(ii) $$R=P+Q=30,$$

and $$11.AC=19.BC=19(2\tfrac{1}{2}-AC),$$

whence $$AC=1\tfrac{7}{12}\text{ ft.}=1\text{ ft. }7\text{ ins.}$$

(iii) $$R=P+Q=10,$$

and $$AC=\frac{1}{2}AB=1\text{ ft. }6\text{ ins.}$$

2. (i) $$R = Q - P = 8,$$
and $$17 \,.\, AC = 25 \,.\, BC = 25\,(AC - 8),$$
whence $$AC = 25 \text{ ins.}$$
(ii) $$R = P - Q = 8,$$
and $$23 \,.\, AC = 15 \,.\, BC = 15\,(AC - 40)$$
whence $$AC = -75 \text{ ins}$$
(iii) $$R = P - Q = 17,$$
and $$26 \,.\, AC = 9 \,.\, BC = 9\,(AC - 36),$$
whence $$AC = -19\tfrac{1}{17} \text{ ins.}$$

3. (i) $$Q = R - P = 9,$$
and $$8 \,.\, 4\tfrac{1}{2} = 9 \,.\, BC,$$
so that $$BC = 4 \text{ ins.},$$
and $$AB = 4\tfrac{1}{2} + 4 = 8\tfrac{1}{2} \text{ ins.}$$
(ii) $$P \,.\, 7 = 11\,(8\tfrac{3}{4} - 7),$$
whence $$P = 2\tfrac{3}{4},$$
and $$R = P + Q = 13\tfrac{3}{4}.$$
(iii) $$6 \,.\, 9 = Q \,.\, 8,$$
so that $$Q = 6\tfrac{3}{4},$$
and $$R = P + Q = 12\tfrac{3}{4}.$$

4. (i) Since AC is positive, Q is $> P$;
hence $$Q = P + R = 25,$$
and $$8 \,.\, 4\tfrac{1}{2} = 25 \,.\, BC,$$
whence $$BC = \frac{36}{25} \text{ in.}$$
$$\therefore\ AB = AC - BC = 3\tfrac{3}{50} \text{ ins.}$$
(ii) $$BC = 8\tfrac{3}{4} + 7 = 15\tfrac{3}{4} \text{ ins.},$$
so that $$P \,.\, 7 = 11 \,.\, 15\tfrac{3}{4},$$
whence $$P = 24\tfrac{3}{4},$$
and $$R = P - Q = 13\tfrac{3}{4}.$$
(iii) $$BC = 12 + 9 = 21 \text{ ins.},$$
$$\therefore\ 6 \,.\, 9 = Q \,.\, 21,$$
whence $$Q = 2\tfrac{4}{7},$$
and $$R = P - Q = 3\tfrac{3}{7}.$$

5. Let P lbs. wt. and Q lbs. wt. be the required forces acting at distances of 6 ins. and 18 ins. respectively from the given force.

Then $$P + Q = 20;$$
also $$6P = 18Q, \text{ i.e. } P = 3Q;$$
$$\therefore\ P = 15 \text{ lbs. wt., and } Q = 5 \text{ lbs. wt.}$$

6. Let P lbs. wt. (the greater) and Q lbs. wt. be the required forces acting at distances of 8 ins. and 26 ins. respectively from the force of 30 lbs. wt.

Then $$P-Q=30, \text{ and } 8P=26Q;$$

$$\therefore\ 4P=13(P-30), \text{ whence } P=43\tfrac{1}{3} \text{ lbs. wt.,}$$

and, therefore, $$Q=13\tfrac{1}{3} \text{ lbs. wt.}$$

7. If A and B be the points of application of the forces P and Q, and C be the point in which their resultant R meets AB, then

$$P.AC=Q.BC;$$

if $\dfrac{P^2}{Q}$ be put for Q, we have $P.AC=\dfrac{P^2}{Q}.BC$,

$$Q.AC=P.BC. \qquad \text{Q.E.D.}$$

8. Let AB be the pole and R and S be the pressures at the points of support. Let C be the point at which the cask is suspended, so that $AC=3\frac{1}{2}$ feet and $CB=2\frac{1}{2}$ feet.

Then $$R+S=1\tfrac{1}{2} \text{ cwt.}=168 \text{ lbs.}$$

Also $$R\times 3\tfrac{1}{2}=S\times 2\tfrac{1}{2},$$

$$\therefore\ 7R=5S=5(168-R),$$

whence $$R=70 \text{ lbs. wt.,}$$

and, therefore, $$S=98 \text{ lbs. wt.}$$

9. Let AB be the plank, and let the stronger man be at A. Then, if Q lbs. be the weight supported by the other man at B, and the block of 270 lbs. be at distance of x feet from A, we have

$$Q=270-180=90 \text{ lbs.};$$

hence $$180x=90(6-x), \text{ whence } x=2 \text{ feet.}$$

10. Let AB be the rod, and C be its middle point. If the required point D be at distance of x feet from A, where the weight of 7 lbs. is hung, then, since $AC=6$ feet, we have $7x=17(6-x)$, whence

$$x=4\tfrac{1}{4} \text{ feet.}$$

11. Let AB be the rod, and C be its middle point where the required weight W lbs. acts. Let D be the point about which the rod balances when the load of 5 lbs. is placed at A, so that $AD=3$ ins., and $DC=15$ ins.

Then $$5\times 3=W\times 15, \text{ whence } W=1 \text{ lb.}$$

12. Let AB be the bar, and C be its middle point where its weight 3 lbs. acts. Let D be the prop at distance of x feet from C. Then, if the vertical force of 1 lb. wt. act at A, we have

$$1.AD=3.CD,$$

$$(2+x)=3x, \text{ whence } x=1 \text{ foot.}$$

13. Let W lbs. be the weight of the rod acting at C its middle point, and let x feet and y feet be the distances of the pegs from C in the two cases respectively; then we have

$$10(2-x)=Wx, \text{ and } 4(2-y)=Wy;$$

also
$$x+y=1.$$

Hence, by division,

$$\frac{5(2-x)}{2(2-y)}=\frac{x}{y}, \ i.e. \ \frac{10-5x}{2+2x}=\frac{x}{1-x},$$

whence
$$x=\frac{2}{3}\text{ ft.}=8\text{ ins., and } y=\frac{1}{3}\text{ ft.}=4\text{ ins.}$$

$$\therefore\ W=\left[10\left(2-\frac{2}{3}\right)\right]\div\frac{2}{3}=20\text{ lbs.}$$

14. If CDE be the rod, D be its middle point, and P lbs. wt. and Q lbs. wt. be the pressures on the rails at A and B respectively, (A being the point near C), then

$$P+Q=8, \text{ and } P-Q=6;$$

whence
$$P=7\text{ lbs. wt., and } Q=1\text{ lb. wt.}$$

Also, by taking moments about D, we have

$$7.AD=1.BD,$$

i.e.
$$7AD=5-AD, \qquad \text{so that } AD=\frac{5}{8}\text{ in.};$$

but
$$CD=DE=15\text{ ins.},$$

so that
$$CA=CD-AD=14\tfrac{3}{8}\text{ ins.},$$

and
$$EB=DE-BD=DE-7AD=10\tfrac{5}{8}\text{ ins.}$$

15. If $ACDB$ be the beam, C be its middle point, and P and Q be the pressures on the props at A and D respectively, moments about C give $P.AC=Q.CD$; but $AD=3$ ft., and $AC=2$ ft., so that $CD=1$ ft., and hence $2P=Q$.

16. Let ABC be the rod resting under the peg at A and over the peg at B. Then $AB=3$ ins., and $BC=21$ ins. If P lbs. wt. (downwards at A) and Q lbs. wt. (upwards at B) be the required pressures, we have $P+5=Q$. Also, moments about A give $3Q=5\times 24$;

$$\therefore\ Q=40\text{ lbs. wt., and } P=Q-5=35\text{ lbs. wt.}$$

17. Let AB be the rod, A being the point in contact with the horizontal plane, and let C be its middle point. The rod is in equilibrium under the action of the three forces, its weight W acting vertically downwards through C, R the reaction at A acting vertically upwards, and T the tension of the string. Since the directions of two of them (W and R) are parallel, the direction of T must be parallel to the directions of W and R. Also, since W balances the resultant of R and T, we see that T must act upwards; and AC being equal to CB, T must be equal to R; but $T+R=W$, and hence $T=\dfrac{W}{2}$.

18. Let AB be the stick, A being the position of the hand. Then, if W be the weight of the bundle at B, R be the pressure on the shoulder, l be the whole length of the stick, and x be the distance between the hand and the shoulder, moments about A give

$$R\times x=W\times l, \qquad \text{so that } R=\frac{Wl}{x}.$$

Hence R varies inversely as x.

19. Referring to the last example, we have, if P lbs. wt. be the force exerted by the hand, and R lbs. wt. be the pressure on the shoulder,

$$\left.\begin{array}{ll}(1) & R-P=50,\\ \text{and} & R\times 12=50\times 36,\end{array}\right\}\ \begin{array}{ll}\text{whence} & R=150 \text{ lbs. wt.},\\ \text{and} & P=100 \text{ lbs. wt.};\end{array}$$

$$\left.\begin{array}{ll}(2) & R-P=50,\\ \text{and} & R\times 18=50\times 36,\end{array}\right\}\ \begin{array}{ll}\text{whence} & R=100 \text{ lbs. wt.},\\ \text{and} & P=50 \text{ lbs. wt.};\end{array}$$

$$\left.\begin{array}{ll}(3) & R-P=50,\\ \text{and} & R\times 24=50\times 36,\end{array}\right\}\ \begin{array}{ll}\text{whence} & R=75 \text{ lbs. wt.},\\ \text{and} & P=25 \text{ lbs. wt.}\end{array}$$

20. Let $ECBA$ be the bar, the right-hand force acting at A and the other two at B and C, so that $CB=1$ ft., and $BA=2$ ft. The resultant of 1 lb. wt. at B and 1 lb. wt. at C is 2 lbs. wt. at D the middle point of CB; the resultant of 2 lbs. wt. at D and 1 lb. wt. at $A=(2-1)=1$ lb. wt. at E,

where $$2\times DE=1\times AE,$$

i.e. $$2(AE-2\tfrac{1}{2})=AE;$$

$$\therefore\ AE=5 \text{ feet.}$$

21. Let BC be the lower edge of the portmanteau, B being lower than C, and O its centre of gravity; let the vertical through O and the perpendicular from O on BC meet BC in L and K respectively. Then

$$\angle KOL=30^\circ, \text{ and } LK=OK\tan 30^\circ=\frac{1}{\sqrt{3}} \text{ feet.}$$

$$\therefore\ \frac{BL}{LC}=\frac{BK-LK}{BK+LK}=\frac{\frac{3}{2}-\frac{1}{\sqrt{3}}}{\frac{3}{2}+\frac{1}{\sqrt{3}}}=\frac{3\sqrt{3}-2}{3\sqrt{3}+2}.$$

If R and S be the forces required at B and C, we have

$$R+S=112 \text{ and } R.BL=S.CL;$$

$$\therefore\ \frac{R}{S}=\frac{3\sqrt{3}+2}{3\sqrt{3}-2}. \qquad \therefore\ \frac{R}{112}=\frac{3\sqrt{3}+2}{6\sqrt{3}}.$$

$$\therefore\ R=\tfrac{112}{18}(9+2\sqrt{3})=77\cdot 55 \text{ lbs. wt.},$$

and $$S=112-R=34\cdot 45 \text{ lbs. wt.}$$

EXAMPLES. IX. (Pages 71—74.)

1. The perpendicular from C on $DB = 2\sqrt{2}$ feet. The sum of the moments required

$$= 5 \times 2\sqrt{2} + 4 \times 0 - 3 \times 4 + 2 \times 4 = 10\sqrt{2} - 4$$
$$= 14 \cdot 14 - 4 = 10 \cdot 1 \text{ ft.-lbs.}$$

2. The sum of the moments required

$$= 1 \times 0 - 2 \times \sqrt{3} - 3 \times 2\sqrt{3} + 4 \times 2\sqrt{3} + 5 \times \sqrt{3} + 6 \times 0$$
$$= 13\sqrt{3} - 8\sqrt{3} = 5\sqrt{3} \text{ ft.-lbs.}$$

3. Let AB be the pole, A being the upper end to which the string is attached; and let P lbs. wt. be the required force applied at a point 4 feet above B. Moments about B give

$$P \times 4 = 30 \times 20 \cos 30^\circ = 300\sqrt{3};$$
$$\therefore P = 75\sqrt{3} = 129 \cdot 9 \text{ lbs. wt.}$$

4. Let AB be the rod, C be its middle point where its weight 9 lbs. acts; and let the masses of 6 lbs. and 12 lbs. be suspended at A and B respectively. Then, if D be the required point and $AD=x$ feet, moments about D give $6x+9(x-3)=12(6-x)$,

whence $$x=3\tfrac{2}{3}\text{ ft.}=3\text{ ft. }8\text{ ins.}$$

5. As in the last example, with the bodies of weights 20 lbs. and 30 lbs. at A and B respectively, and the weight of the beam, 50 lbs., at C, we have (D being the required point, and $AD=x$ feet)

$$20x+50(x-6)=30(12-x),\text{ whence } x=\frac{66}{10}=6\cdot6\text{ feet.}$$

6. If x feet be the distance of the point from the end from which the given distances are measured, moments about the point give

$$4(4-x)+3(3-x)=2(x-2)+1(x-1)+4(x-2\tfrac{1}{2}),$$

whence $$x=2\tfrac{6}{7}\text{ feet.}$$

7. If W lbs. be the required weight, moments about the point give $10\times1=2\times1+W\times3$,

whence $$W=2\tfrac{2}{3}\text{ lbs.}$$

8. If W lbs. be the required mass, moments about the point give $6\times4=10\times1+W\times6$,

whence $$W=2\tfrac{1}{3}\text{ lbs.}$$

9. If P tons' wt. and Q tons' wt. be the pressures on the supports,

(1) $$P+Q=8\text{ tons};$$

and moments about the centre of the roadway give $P=Q$;

whence $$P=Q=4\text{ tons' wt.}$$

(2) $$P+Q=8\text{ tons};$$

and moments about the end at which the pressure is Q give

$$P\times30=6\times15+2\times10,$$

whence $P=3\tfrac{2}{3}$ tons' wt. and, therefore, $Q=4\tfrac{1}{3}$ tons' wt.

10. Let B be at a distance of x inches from one of the pegs, and P be the pressures. Then

$$2P=2W+3W=5W,$$

and moments about B give

$$Px+P(10+x)=2W\times20,$$

so that $$2Px+10P=40W,$$

i.e. $$5Wx+25W=40W,$$

whence $$x=3\text{ ins.}$$

11. Let AB be the rod with equal weights W attached to it at distances of 9 ins. from A and 15 ins. from B, and let T_1 and T_2 be the tensions of the strings at A and B respectively. Taking moments about B, we have $T_1 \times 36 = W \times 27 + W \times 15$, whence

$$W = \frac{6}{7} T_1;$$

hence, if $T_1 = 1$ cwt., $W = \frac{6}{7}$ cwt.; and since $T_2 < T_1$, the string at B will not break.

12. Let AB be the beam, C be its middle point where its weight, 40 lbs., acts, and D be the point where the body must be placed. The weight of the beam causes a tension of 20 lbs. wt. in each string. If the mass of 20 lbs. cause a tension of 15 lbs. wt. in the string at A (so that the maximum tension of 35 lbs. wt. be attained in that string) and a tension of 5 lbs. wt. in the string at B, then we have

$$AD : DB = 5 : 15 = 1 : 3;$$

$$\therefore AD : AB = 1 : 4; \text{ hence } AD = \frac{1}{4} AB, \text{ and } CD = \frac{1}{4} AB.$$

13. Let P lbs. wt. be the required force. The bar will just begin to turn about B; hence, avoiding the reaction at B by taking moments about it, we have

$$P \times 10 = 50 \times 5 + 100 \times 3,$$

whence $$P = 55 \text{ lbs. wt.}$$

14. Let AB be the rod, and W lbs. be its weight acting at G the required point; also let C be the middle point of AB, and D and E be the pegs near A and B respectively. Then, if $BG = x$ ins., and the mass of 4 lbs. be at A, moments about D give

$$4 \times 3\tfrac{1}{2} = W(12\tfrac{1}{2} - x);$$

and if the mass of 5 lbs. be at B, moments about E give

$$5 \times 3\tfrac{1}{2} = W(x - 3\tfrac{1}{2});$$

whence $x = 8\frac{1}{2}$ ins., and $W = 3\frac{1}{2}$ lbs., *i.e.* the point G is at a distance of $8\frac{1}{2}$ ins. from the mass of 5 lbs.

15. Moments about the hinge give, W being the required mass in oz.,

$$W \times 24 = 9 \times 8,$$

whence $$W = 3 \text{ oz.}$$

16. If P be the pressure on the ground of each front wheel, and Q be that of the small wheel, in lbs. wt., moments about the axle of the small wheel give

$$2P \times 36 = 126 \times 33 + 74 \times 27,$$

whence $$P = 85\tfrac{1}{2} \text{ lbs. wt.}$$

Also, resolving vertically, $$2P + Q = 200,$$

whence $$Q = 29 \text{ lbs. wt.}$$

17. If P be the pressure on each hind wheel, and Q be that on the front wheel, in lbs. wt., moments about the axle of the hind wheels give

$$Q \times 42 = 84 \times 12 + 154 \times 6,$$

whence $\quad Q = 46$ lbs. wt.

Also, resolving vertically, $\quad 2P + Q = 238$ lbs.,

whence $\quad P = 96$ lbs. wt.

18. Let B be the point where the pressure of 10 lbs. wt. acts, and A be the axle, so that $AB = 6$ feet. Then, if C be the required position, moments about A give

$$4 \times 112 \times AC = 10 \times 6,$$

whence $\quad AC = \dfrac{15}{112}$ ft. $= 1\frac{17}{28}$ ins.

19. The forces of 6 lbs. wt. and 4 lbs. wt. are equal to a parallel force of 2 lbs. wt. at E, in CD produced, such that $DE = 2CD$. The forces of 5 lbs. wt. and 3 lbs. wt. are equal to a parallel force of 2 lbs. wt. at F, in AD produced,

such that $\quad DF = \dfrac{3}{2}AD.$

Hence the required resultant is $2\sqrt{2}$ lbs. wt., parallel to CA, and it cuts AD at P, where

$$AP = AD + DF + FP$$
$$= AD + \frac{3}{2}AD + 2AD = \frac{9}{2}AD.$$

20. P along AB and P along AD have resultant along AC which can be transferred to C and resolved into P along BC and P along DC; hence the required resultant is $2P$ along DC.

21. Since the forces are like, the resultant of the forces proportional to 1 and 4 is proportional to 5 parallel to AB, and cuts AD in F so that $AF : FD = 4 : 1$, *i.e.* $AF = \frac{4}{5}AD = \frac{4}{5}$ ft. Also the resultant of the forces proportional to 2 and 3 is proportional to 5 parallel to AD, and cuts AB in E so that $AE : EB = 2 : 3$, *i.e.* $AE = \frac{2}{5}AB = \frac{2}{5}$ ft. If G be the point of intersection of the above resultants, we have a force proportional to 5 along EG, and a force proportional to 5 along FG. Hence the required resultant is a force proportional to $5\sqrt{2}$, and its direction bisects the angle EGF, *i.e.* is parallel to AC. Also, if its direction cut AD in P, then

$$AP = AF - PF = AF - FG = AF - AE$$
$$= \left(\frac{4}{5} - \frac{2}{5}\right) \text{ ft.} = \frac{2}{5} \text{ ft.}$$

22. If the resultant cut AB at E, the sum of the moments about E is zero;

$$\therefore\ 40 \times BE = 50 \times 4,$$

$$i.e.\ BE = 5 \text{ feet, and } AE = 8 \text{ feet.}$$

Similarly, if the resultant cut AD at F,

$$50 \times DF = 30\,(4 + DF) + 40 \times 8,$$

$$\therefore\ DF = 12 \text{ feet, and } AF = 16 \text{ feet.}$$

The component parallel to AF is 40, and the component parallel to AE is $(50 - 30)$, *i.e.* 20;

$$\therefore \text{ the resultant} = \sqrt{(40)^2 + (20)^2} = 20\sqrt{5} \text{ lbs. wt.}$$

23. If the resultant meet BC in Q, the sum of the moments about Q is zero.

$$\therefore\ P \times BQ \sin 60^\circ = 3P \times CQ \sin 60^\circ.$$

$$\therefore\ BQ = 3\,(BQ - BC),\ i.e.\ BQ = \frac{3}{2}BC.$$

The components of the forces perpendicular to BC

$$= 3P \sin 60^\circ - P \sin 60^\circ = P\sqrt{3}.$$

Also the components parallel to BC

$$= 2P - 3P \cos 60^\circ - P \cos 60^\circ = 0.$$

Hence the resultant is $P\sqrt{3}$, perpendicular to BC, and cutting BC at Q, where $BQ = \frac{3}{2}BC$.

24. The resultant of 1 lb. wt. along AB and 1 lb. wt. along AC acting at an angle of $120^\circ = 2 \cos 60^\circ = 1$ lb. wt. along AQ, where Q is the middle point of BC. Also, since the resultant of the forces along AB, AC bisects BC (and, therefore, meets the force along BC in the middle point of BC), therefore the resultant of all three forces must bisect BC. The resultant of 1 lb. wt. along AQ and $\sqrt{3}$ lb. wt. along $QC = \sqrt{1^2 + (\sqrt{3})^2} = 2$ lbs. wt., and makes the angle $\tan^{-1}\frac{1}{\sqrt{3}}$, *i.e.* 30°, with QC, and is, therefore, parallel to AC.

25. The sum of the moments about X must be zero;

$$\therefore\ 2CA \,.\, CX \sin C = AB \,.\, BX \sin B,$$

or

$$2CX = BC + CX,$$

$$(\text{since } CA \sin C = AB \sin B),$$

so that

$$CX = BC.$$

The components of the forces perpendicular to BC

$$= 2CA \sin C - AB \sin B = CA \sin C.$$

Also the components in the direction CB

$$= 2CA \cos C + AB \cos B - BC = CA \cos C.$$

Hence the resultant $= \sqrt{(CA \sin C)^2 + (CA \cos C)^2} = CA$, and is parallel to CA.

26. If AD, BE and CF meet in G, the given forces are represented by AD, BG and GF, *i.e.* by AG, GD, BG and GF. Forces represented by AG and BG have resultant represented by $2 . FG$. Hence the system reduces to GD and FG, *i.e.* to GD and $\frac{1}{2}GC$. Draw GK parallel to EC; then

$$DK : KC = DG : GA = 1 : 2 = \frac{1}{2} : 1,$$

so that
$$DK . 1 = KC . \frac{1}{2}.$$

Hence, by Art. 42, the resultant of GD and $\frac{1}{2}GC$

$$= \frac{3}{2}GK = \frac{1}{2}AC.$$

Also, by similar triangles,

$$BK : KC = BG : GE = 2 : 1.$$

27. If ABC be the triangle, I be the centre of the incircle, and P, Q, and R be the forces along the sides BC, CA, and AB respectively, then $R = P + Q$.

Hence the sum of the moments of the forces about I, whose perpendicular distance from each force is r,

$$= P . r + Q . r - R . r = 0.$$

Hence (Art. 65, Cor.) the resultant of the forces passes through I.

28. If T be the tension of the telegraph wire, the resultant pull on the post $= 2T \cos 30° = T\sqrt{3}$. Hence, if T' be the tension of the supporting wire, and l be the length of the post, moments about the foot of the post give

$$T' \times \frac{l}{2} \sin 30° = T\sqrt{3} \times l,$$

$$\text{whence } T' = 4T\sqrt{3}.$$

29. Let AB be the pillar, A being the base and B the point to which the rope CB must be fixed. From A draw AD perpendicular to BC the direction of the given force, (F), and let the angle $BCA = \theta$. The moment of F about A

$$= F \times AD = F \times AB \cos\theta = F \times l \sin\theta \cos\theta = \frac{F}{2} \times l \sin 2\theta,$$

where l is the length of the rope;

this is greatest when $2\theta = 90°$, *i.e.* $\theta = 45°$, and then

$$AB = BC \sin 45° = \frac{1}{2} l\sqrt{2}.$$

30. Dividing the known moments of the force by the magnitude of the force, we obtain the perpendicular distances of its direction from A and B; with centres A and B, and radii equal to these distances, describe circles. The line of action required is a common tangent to these circles. There are four solutions according to the signs of the given moments. If these be of the same sign, either external tangent is the line required. If of opposite signs, an internal tangent must be taken. If the moments be given of opposite signs and the circles intersect, there is no solution.

31. If the lines representing the forces meet in A, as a force may be supposed to act at any point in its line of action, suppose the two forces represented by AB and AC, and let P be any point about which their moments in the same sense are equal; then the triangle $PAB=$ the triangle PAC; but these triangles are on the same base AP; they must therefore be between the same parallels, *i.e.* AP is parallel to BC, so that the locus of P is the straight line through A parallel to BC. If CAD be the angle exterior to the angle BAC, then the $\angle DAP=$ the $\angle ABC$ and the $\angle PAC=$ the $\angle ACB$;

$$\therefore \sin DAP : \sin PAC = \sin ABC : \sin ACB = AC : AB,$$

i.e. in the inverse ratio of the forces.

If the forces be parallel in the same direction, the locus is a straight line outside the larger one; if parallel in opposite directions, the locus is a parallel straight line between the lines of action at distances from them inversely proportional to the forces.

32. Since an angle in a semicircle is a right angle, we have AP and AQ perpendicular to PB and QB, respectively. Hence $APBQ$ is a rectangle, and therefore $AP=BQ$, and $BP=AQ$. The moment of BP about $A=BP \times AP$, the moment of BQ about $A=BQ \times AQ$; these are equal.

33. Taking moments about the centre of the crank axle, we have

$$T \times 4 = 150 \times 6, \text{ so that } T = 225 \text{ lbs. wt.}$$

EXAMPLES. X. (Page 79.)

1. $5\sqrt{2}$ lbs. wt. acting along AC is equivalent to 5 lbs. wt. along AB and 5 lbs. wt. along AD; $2\sqrt{2}$ lbs. wt. along DB is equivalent to 2 lbs. wt. along DA and 2 lbs. wt. along DC; hence the system is equivalent to 6 lbs. wt. along AB and 6 lbs. wt. along CD together with 2 lbs. wt. along DA and 2 lbs. wt. along BC. These give two like couples, of moments 12 ft.-lbs. and 4 ft.-lbs., which, by Art. 72, have as resultant a couple of moment 16 ft.-lbs.

2. Two unlike couples of moments 5×3 and 2×3, *i.e.* 15 ft.-lbs. and 6 ft.-lbs., give as resultant a couple of moment 9 ft.-lbs.

3. Let a be the distance between parallel sides of the hexagon. Three couples of moments $+11a$, $-5a$, $-xa$ are in equilibrium; hence $11a-5a-xa=0$, whence $x=6$.

4. The couple $1.AB$ requires another couple to balance it; therefore the required force is equal, parallel, and opposite to the force of 5 lbs. wt. at C, acting at a point C' in AC. The moments of these couples must be equal, so that

$$5.\,CC'\sin 30^\circ = 1.\,AB;$$

$$\therefore\ CC' = \frac{2}{5}AB.$$

Page 88. *Numerical Example.* We have

$$a+b=5,\quad x+y=7,\quad \text{and}\quad \frac{x}{y}=\frac{a}{b}=\frac{4}{3};$$

$$\therefore\ x=4\text{ ft., and } y=3\text{ ft.};$$

and, since $5^2=4^2+3^2$, the angle AOB is a right angle;

$$\cos\theta=\frac{l\sin a}{a+b}=\left(7\times\frac{1}{\sqrt{2}}\right)\div 5=\frac{7}{5\sqrt{2}}.$$

$$\therefore\ \tan\theta=\frac{1}{7},\ i.e.\ \theta=\tan^{-1}\frac{1}{7}.$$

Also $$T=\frac{W}{2\cos a}=\frac{W}{2\cos 45^\circ}=W\times\frac{\sqrt{2}}{2},$$

i.e. $$T : W=\sqrt{2} : 2.$$

Page 90. *Numerical Example.* We have

$$W=40\text{ lbs.},\quad 2a=90^\circ,\quad \text{and}\quad a : b=1 : 2;$$

$$\therefore\ \tan\theta=\frac{b-a}{b+a}\tan a=\frac{2-1}{2+1}\times 1=\frac{1}{3}.$$

Also $$\frac{R}{\sin(a+\theta)}=\frac{S}{\sin(a-\theta)}=\frac{W}{\sin 2a}$$

gives $$\frac{R\sqrt{2}}{\cos\theta+\sin\theta}=\frac{S\sqrt{2}}{\cos\theta-\sin\theta}=\frac{40}{1},$$

i.e. $$\frac{R\sqrt{20}}{3+1}=\frac{S\sqrt{20}}{3-1}=40,$$

whence $R=16\sqrt{5}$ lbs. wt., and $S=8\sqrt{5}$ lbs. wt.

EXAMPLES. XI. (Pages 92—96.)

1. W acts vertically through G, the middle point of AB; and if the vertical line through G meet BC in D, D is the middle point of BC. Since the directions of P and W pass through D, for equilibrium the direction of the reaction at A must also pass through D. The angle ADC is a right angle, and the triangle CAD is the triangle of forces. Hence we have

$$\frac{P}{DC}=\frac{W}{CA}, \text{ i.e. } \frac{P}{W}=\frac{DC}{CA}=\cos ACB,$$

i.e. $$P=W\cos ACB.$$

2. [Cf. the figure p. 85.] Let AB be the rod, W be its weight acting at C its middle point and $F\left(=\frac{W}{2}\right)$ be the horizontal force acting at B. From A and C draw vertical lines AE and CD meeting the horizontal line through B in E and D respectively. Join DA. Since the directions of F and W meet at D, for equilibrium the direction of the reaction at A must be in the line DA. The triangle AED is the triangle of forces, and we have

$$\frac{F}{ED}=\frac{W}{AE}, \text{ i.e. } ED=\frac{1}{2}AE;$$

hence also we have

$$ED=\frac{1}{2}EB; \text{ i.e. } AE=EB,$$

and $$\tan\theta=\tan EAB=\frac{EB}{AE}=1, \text{ i.e. } \theta=45°.$$

Otherwise thus: Taking moments about A, we have, if the angle EAB be θ,

$$F.AB\cos\theta=W.AC\sin\theta,$$

whence $$\tan\theta=1, \text{ i.e. } \theta=45°.$$

3. Since the mass of 10 lbs. and the tension T of the string along BC act through B, for equilibrium the reaction R at the hinge A must also act through B; hence it acts along AB. Resolving vertically and horizontally, we have

$$T\cos 45°=10, \text{ and } R=T\cos 45°,$$

whence $$T=10\sqrt{2} \text{ lbs. wt., and } R=10 \text{ lbs. wt.}$$

Otherwise thus: With the triangle ABC as the triangle of forces, we have

$$\frac{T}{BC}=\frac{R}{AB}=\frac{10}{CA},$$

$$\frac{T}{\sqrt{2}}=\frac{R}{1}=\frac{10}{1}, \text{ whence } T \text{ and } R.$$

4. Let D be the point in the wall to which the string is fastened. For equilibrium, the lines of action of W the weight of the rod (acting at G its middle point), R the horizontal reaction at A, and T the tension of the string must pass through a point O. Then AOG is a right-angled triangle, and C is the middle point of the hypotenuse AG; therefore $OC=AC=CG$. Also the angle $COA=$ the angle CAO. Hence, by Euc. I. 26, the triangles AOG and OAD are identically equal, therefore $OD=AG=\frac{1}{2}AB$; but $OC=AC=\frac{1}{4}AB$, and therefore

$$CD=\frac{1}{4}AB.$$

5. For equilibrium, the lines of action of W, R the horizontal reaction at A, and T the tension of CD must pass through a point, O. The triangle DAO is the triangle of forces, and we have

$$\frac{T}{OD}=\frac{R}{AO}=\frac{W}{DA},$$

whence $$T=W\sec 30^\circ=\frac{2}{3}W\sqrt{3},$$

and $$R=W\tan 30^\circ=\frac{1}{3}W\sqrt{3}.$$

6. For equilibrium, the lines of action of W the weight of the rod, R the horizontal reaction at A, and T the tension of the string BC must pass through a point, O. The triangle ACO is the triangle of forces, and we have

$$\frac{T}{OC}=\frac{W}{CA}=\frac{R}{AO};$$

$$\therefore\ T>W, \text{ since } OC>CA.$$

[Cf. the figure p. 86, putting C for D, and with the string attached at B.]

7. First let the angle BAD be acute. Draw AD vertical and AB equal to the given length of beam. Let the vertical through G, the middle point of AB, and the horizontal through A meet at the point E. Join E to the given point C on the beam, and produce EC to D. Then we have given, graphically, CD the length of the string, and AD, fixing the position of D. As the angle BAD increases towards a right angle, G more and more nearly coincides with E, and ultimately in the horizontal position of AB, the triangle GCE vanishes into a point, *i.e.* C has to be at G, wherever D may be. When the angle BAD is obtuse, C is necessarily below E, yet CE must still meet the wall at D above A. Hence, obviously C lies further from the wall than G in this case. If D and A remain constant throughout, clearly C moves away from A as the angle BAD increases.

8. Take a figure as on p. 86, with the string BOD attached at B, BEF horizontal to meet the wall at F, and the angle $BAF=\theta$. The three forces acting on the rod pass through a point O. Since

$$AG=GB, \text{ therefore } BF=2AO,$$

so that
$$FA=AD;$$

also
$$FA=a\cos\theta, \text{ and } BF=a\sin\theta;$$

but
$$DF^2+BF^2=DB^2,$$

so that
$$4a^2\cos^2\theta+a^2\sin^2\theta=l^2,$$

i.e.
$$3a^2\cos^2\theta+a^2=l^2, \text{ and } \cos^2\theta=\frac{l^2-a^2}{3a^2}.$$

Algebraically, for real values of $\cos\theta$, $l>a$; and, since $\cos\theta$ is not >1, $l^2-a^2<3a^2$, *i.e.* $l^2<4a^2$, *i.e.* $l<2a$; hence $\frac{a}{l}$ must be <1 and $>\frac{1}{2}$.

9. [Draw a figure as on p. 87, with C for O, P for T, and G for C.] Since W balances the resultant of two equal forces P, CA and CB are equally inclined to the vertical; hence

$$\text{the } \angle ACG=\text{the } \angle BCG=\alpha, \text{ say.}$$

Let θ be the inclination of AB to the horizon. Resolving vertically, we have

$$2P\cos\alpha=W, \ i.e.\ \alpha=\cos^{-1}\frac{W}{2P}.$$

The first theorem of Art. 79 then gives

$$(a+b)\cot BGC=a\cot\alpha-b\cot\alpha;$$

also
$$\angle BGC=90^\circ-\theta;$$

whence

$$\tan\theta=\frac{a-b}{a+b}\cot\alpha=\frac{a-b}{a+b}\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right),$$

i.e.
$$\theta=\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

10. Let AB be the beam, C be the fixed point, and CA and CB be the strings. Take $AC : BC : AB = 2 : 3 : 4$. The weight of the beam, W, acts at G its middle point, and G must be vertically under C, since the resultant of the tensions of CA and CB (T_1 and T_2, say) balances W. Complete the parallelogram of which CA and CB are adjacent sides: since G is the middle point of AB, the straight line CG produced (to D, say) must be the diagonal of the parallelogram. Then the triangle CBD is the triangle of forces. Now

$$2(CG^2 + AG^2) = AC^2 + CB^2,$$

i.e.
$$2(CG^2 + 4) = 4 + 9 = 13,$$

whence
$$2CG^2 = 5;$$

$$\therefore CD^2 = 4CG^2 = 10, \text{ and } CD = \sqrt{10};$$

hence
$$T_1 : T_2 : W = AC : AD : CD = 2 : 3 : \sqrt{10}.$$

11. Let AB be the rod, G be its middle point, C be the fixed point, and AC and BC be the strings, so that

$$AB = 15 \text{ ins., } AC = 12 \text{ ins., and } BC = 9 \text{ ins.}$$

Then G is vertically below C, and since $(15)^2 = (12)^2 + 9^2$, the angle ACB is a right angle. Hence we have

$$GC = GA = GB = \frac{15}{2},$$

and
$$\sin CGB = \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\sin CAB\cos CAB = 2 \cdot \frac{CB}{AB} \cdot \frac{AC}{AB} = 2 \cdot \frac{9}{15} \cdot \frac{12}{15} = \frac{24}{25};$$

$$25\sin\theta = 24.$$

12. Let AB be the rod, G be its middle point, C be the peg, and T_1 and T_2 be the tensions of the strings AC (the longer) and BC respectively. Then G is vertically below C; and since $AG = GB$ and ACB is a right angle, $GC = GA = GB$, and therefore

$$\text{the } \angle CAG = \text{the } \angle GCA = \theta,$$

and
$$\text{the } \angle GCB = \text{the } \angle GBC = \frac{\pi}{2} - \theta.$$

Also, since
$$AC = 2BC, \tan\theta = \frac{BC}{AC} = \frac{1}{2},$$

and
$$\sin\theta = \frac{1}{\sqrt{5}}, \text{ and } \cos\theta = \frac{2}{\sqrt{5}}.$$

By Lami's Theorem, we have

$$\frac{T_1}{\sin GCB}=\frac{T_2}{\sin GCA}=\frac{3}{\sin ACB},\ i.e.\ \frac{T_1}{\cos\theta}=\frac{T_2}{\sin\theta}=3;$$

$$\therefore T_1=3\cos\theta=3\cdot\frac{2}{\sqrt{5}}=\frac{6\sqrt{5}}{5}\text{ lbs. wt.},$$

and $$T_2=3\sin\theta=3\cdot\frac{1}{\sqrt{5}}=\frac{3\sqrt{5}}{5}\text{ lbs. wt.}$$

Otherwise thus: Draw GD perpendicular to AC, then CGD is a triangle of forces, since GD is parallel to BC. Let $BC=1$, so that

$$AC=2;$$

$$\therefore CD=1,\ GD=\frac{1}{2},\ \text{and}\ CG=\frac{\sqrt{5}}{2}.$$

Hence we have

$$T_1 : T_2 : 3\text{ lbs.}=1:\frac{1}{2}:\frac{\sqrt{5}}{2}=2:1:\sqrt{5};$$

$$\therefore T_1=\frac{6}{\sqrt{5}}=\frac{6\sqrt{5}}{5}\text{ lbs. wt., and } T_2=\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}\text{ lbs. wt.}$$

13. Let A and B be the centres of the spheres, and O be the centre of the cup; then $AO=BO=2$ ins. $=AB$; hence ABO is an equilateral triangle. Let S be the horizontal reaction between the spheres, and R be the reaction between either sphere and the cup, which passes through A or B and O. Consider one sphere; resolving horizontally, we have

$$S=R\cos 60^\circ=\frac{R}{2},\ i.e.\ R=2S.$$

14. If R and S be the required reactions respectively, then resolving vertically and horizontally, we have

$$W=S\sin\alpha,\ \text{and}\ R=S\cos\alpha;$$

whence $$R=W\cot\alpha,\ \text{and}\ S=W\operatorname{cosec}\alpha.$$

Otherwise thus: By Lami's Theorem, we have

$$\frac{R}{\sin\left(\frac{\pi}{2}+\alpha\right)}=\frac{S}{\sin\frac{\pi}{2}}=\frac{W}{\sin(\pi-\alpha)}.$$

$$\therefore \frac{R}{\cos\alpha}=S=\frac{W}{\sin\alpha},$$

whence $$R=W\cot\alpha,\ \text{and}\ S=W\operatorname{cosec}\alpha.$$

15. Let C be the centre of the sphere, and R and S be the reactions respectively at A and B, the points where the vertical plane through C perpendicular to the bars cuts them. Then ABC is an equilateral triangle; and since AB is horizontal, AC and BC are each inclined at an angle of 30° to the vertical. Hence, resolving horizontally and vertically, W being the weight of the sphere, we have

$$R\cos 60° = S\cos 60°, \text{ and } 2R\cos 30° = W;$$

$$\therefore R = S = \frac{W}{\sqrt{3}} = \frac{1}{3}W\sqrt{3}.$$

16. Let B and D be the points of attachment of the string to the wall and the sphere respectively. The reaction R of the wall at the point of contact A, being horizontal, passes through C the centre of the sphere, the weight W of which also acts through C; therefore the remaining force T, the tension of the string BD, must act in a line through C; therefore BDC is a straight line. Also, since $BD+DC=2AC$, the $\angle ACB = 60°$, and the $\angle ABC = 30°$. The triangle ABC is the triangle of forces. Hence we have

$$\frac{T}{BC} = \frac{R}{AC} = \frac{W}{BA}, \text{ i.e. } \frac{T}{2} = \frac{R}{1} = \frac{W}{\sqrt{3}};$$

$$\therefore T = \frac{2}{3}W\sqrt{3}, \text{ and } R = \frac{1}{3}W\sqrt{3}.$$

Otherwise thus: By Lami's Theorem, we have

$$\frac{T}{W} = \frac{\sin 90°}{\sin 120°} = \frac{2}{\sqrt{3}}, \text{ i.e. } T = \frac{2}{3}W\sqrt{3};$$

and
$$\frac{R}{W} = \frac{\sin 150°}{\sin 120°} = \frac{1}{\sqrt{3}}, \text{ i.e. } R = \frac{1}{3}W\sqrt{3}.$$

Otherwise thus: Resolving vertically and horizontally, we have

$$T\cos 30° = W, \text{ i.e. } T = \frac{2}{3}W\sqrt{3};$$

and
$$R = T\sin 30°, \text{ i.e. } R = \frac{1}{2}T = \frac{1}{3}W\sqrt{3}.$$

17. The tensions of the strings in the two cases are the same throughout; denote them by T and T' respectively. Let W be the weight of the picture, l be the original length of the string, and 2θ be the angle at the peg in the second case. Resolving vertically, we have in the first case $2T\cos 30° = W$; and in the second case $2T'\cos\theta = W$.

Now
$$\sin\theta = \frac{l}{2} \div \frac{2l}{3} = \frac{3}{4},$$

and therefore
$$\cos\theta = \frac{\sqrt{7}}{4};$$

hence we have

$$\frac{T}{T'} = \frac{\cos\theta}{\cos 30°} = \frac{\sqrt{7}}{4} \div \frac{\sqrt{3}}{2} = \frac{\sqrt{7}}{2\sqrt{3}}, \text{ i.e. } T : T' = \sqrt{7} : 2\sqrt{3}.$$

18. The tension of the cord is the same throughout; denote it by T. Resolving vertically, we have

$$2T\cos 30° = 40,$$

whence $$T = \frac{40}{3}\sqrt{3} \text{ lbs. wt.}.$$

19. Here, if 2θ be the angle at the nail, and T be the tension of the string, resolving vertically, we have

$$2T\cos\theta = 10; \text{ but } \cos\theta = 1\tfrac{1}{2} \div 2 = \frac{3}{4};$$

hence $$T = 10 \times \frac{2}{3} = 6\tfrac{2}{3} \text{ lbs. wt.}$$

20. Let $ABCD$ be the vertical section of the picture perpendicular to the wall, through G, its centre of gravity; also let D and E be the points of attachment of the string to the picture and the wall respectively, and A be the point in contact with the wall. Since $DE = DA = a$, therefore DN, perpendicular to the wall, bisects EA in N; and if ED produced meet the horizontal line from A at F, then $AF = 2DN$. The directions of the three forces acting on the picture, viz. its weight vertically downwards through G, the tension of the string, and the horizontal reaction at A, must therefore meet in the point F. Let the $\angle DAN = \theta =$ the $\angle DEN$; then $DN = a\sin\theta$. Draw GH perpendicular to DA, and HK and GL perpendicular to the wall. Then the inclination of GH to the horizon is θ, and $HK = \frac{1}{2}DN$; also, since GF is vertical, $GL = AF$. Hence we have

$$GH\cos\theta + HK = AF, \text{ i.e. } \frac{b}{2}\cos\theta + \frac{a}{2}\sin\theta = 2a\sin\theta;$$

whence $$3a\sin\theta = b\cos\theta, \text{ and } \tan\theta = \frac{b}{3a},$$

i.e. $$\theta = \tan^{-1}\frac{b}{3a}.$$

21. Let AB be the vertical section of the picture perpendicular to the wall, through G its centre of gravity, A being the point in contact with the wall; and let the cord be attached to the point C on the wall. Make the angle CAB equal to a. The weight of the picture acts vertically through G, and the reaction of the wall horizontally through A; if these meet at the point D, the direction of the cord, for equilibrium, must be CD, meeting AB in P, the required point; and CP is the length of the cord. We have $AC = h$; also, if $AB = 2a$, then $AD = a\sin a$, and

$$CD = \sqrt{h^2 + a^2\sin^2 a}.$$

Also, from similar triangles CAP and DGP, we have

$$CP : h = PD : GD = PD : a\cos\alpha = CD : h + a\cos\alpha;$$

hence $$CP = \frac{h}{h + a\cos\alpha}\sqrt{h^2 + a^2\sin^2\alpha}.$$

22. As in Ex. 5, p. 89, $\tan\theta = \dfrac{b-a}{b+a}\tan\alpha$; but

$$a + b = 2r\sin\alpha, \text{ so that } \tan\theta = \frac{b-a}{2r\sin\alpha} \times \frac{\sin\alpha}{\cos\alpha} = \frac{b-a}{2c\cos\alpha};$$

hence

$$\sin\theta = \frac{b-a}{\sqrt{4r^2\cos^2\alpha + (b-a)^2}} = \frac{b-a}{\sqrt{4r^2 + (b-a)^2 - 4r^2\sin^2\alpha}}$$

$$= \frac{b-a}{\sqrt{4r^2 + (b-a)^2 - (a+b)^2}} = \frac{b-a}{2\sqrt{r^2 - ab}}.$$

Again, by Lami's Theorem, we have

$$\frac{R}{\sin(\alpha+\theta)} = \frac{S}{\sin(\alpha-\theta)} = \frac{W}{\sin 2\alpha};$$

$$\therefore R = \frac{W\sin(\alpha+\theta)}{\sin 2\alpha} = W\frac{\frac{b}{r}\cos\theta}{2\sin\alpha\cos\alpha} = W\cdot\frac{b}{r}\cdot\frac{2r\cos\alpha}{2\sqrt{r^2-ab}\,.\,2\sin\alpha\cos\alpha}$$

$$= W\cdot\frac{b}{2\sqrt{r^2-ab}\,.\,\sin\alpha} = \frac{Wbr}{(a+b)\sqrt{r^2-ab}}.$$

Similarly, $$S = \frac{War}{(a+b)\sqrt{r^2-ab}}.$$

23. Cf. the figure p. 90, with the $\angle OGB = \theta$. Here AOB is equilateral, also $AG = \frac{1}{3}AB$. Hence $AG = \frac{1}{2}GB$, so that $\dfrac{AG}{GO} = \dfrac{1}{2}\cdot\dfrac{GB}{GO}$,

i.e. $$\frac{\sin(\theta - 60^\circ)}{\sin 60^\circ} = \frac{1}{2}\cdot\frac{\sin(\theta + 60^\circ)}{\sin 60^\circ},$$

so that $$2\sin(\theta - 60^\circ) = \sin(\theta + 60^\circ).$$

$$\therefore 2\sin\theta\cos 60^\circ - 2\cos\theta\sin 60^\circ = \sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ$$

$$\therefore \sin\theta\cos 60^\circ = 3\cos\theta\sin 60^\circ,$$

whence $$\tan\theta = 3\sqrt{3}, \ \textit{i.e. } \theta = \tan^{-1}(3\sqrt{3}).$$

24. Take the figure p. 89. Here $AD = 2\sqrt{3}$;

$$\therefore AE = 2\sqrt{3}\cos 2\theta; \text{ also } AE = AG\cos\theta = 2\cos\theta.$$

$$\therefore \sqrt{3}\cos 2\theta = \cos\theta, \text{ whence } \theta = 30^\circ,$$

and $$AC = AD\cos 30^\circ = 2\sqrt{3} \times \frac{\sqrt{3}}{2} = 3 \text{ ins.};$$

$$\therefore CB = 1 \text{ inch.}$$

Let $2a$ be the length of the shortest rod. In this case B must coincide with C, and

$$2a = AC = 2OC\cos\theta = 2\sqrt{3}\cos\theta.$$

The result of Ex. 4, Page 88, then gives

$$2\sqrt{3}\cos 2\theta = \sqrt{3}\cos^2\theta, \text{ so that } \cos^2\theta = \tfrac{2}{3},$$

and then $$2a = 2\sqrt{3} \times \frac{\sqrt{2}}{\sqrt{3}} = 2\sqrt{2} \text{ inches}.$$

25. Draw BA at 20° to the horizontal, and through A and B draw straight lines AK, BK at 55° and 50° to the horizontal to meet in K. These lines are the directions of the tensions T and T'; the third force, viz. the weight of the beam, must pass through K. Hence if we draw KG vertical to meet BA in G then G is the centre of gravity; and, on the same scale that AB represents 10 feet, AG represents the distance required.

Draw KL vertically and equal to 4 inches, thus representing the weight on the scale of 50 lbs. to the inch; draw LM parallel to KB to meet AK produced in M; then LM, MK represent the tensions T' and T, on the same scale. On measurement, $LM = 2{\cdot}37$ and $MK = 2{\cdot}66$ ins.

26. Let the vertical through D meet CB in O; then the action at A must pass through O Then COA is the triangle of forces, and therefore

$$\frac{\text{thrust in } CO}{10 \text{ lbs.}} = \frac{CO}{AC}.$$

$$\therefore \text{ Thrust} = \tfrac{10}{6} . CO \text{ lbs. wt.}$$

Also $$\frac{CO}{CB} = \frac{AD}{AB} = \frac{9}{24}, \text{ so that } CO = \frac{3}{8}\sqrt{6^2+24^2}$$

$$= \tfrac{9}{4}\sqrt{17}.$$

$$\therefore \text{ Thrust} = \tfrac{15}{4}\sqrt{17} \text{ lbs. wt.} = \text{etc.}$$

27. Scale—1 inch = 1 foot. Draw DC vertically and equal to 2 inches, and describe a triangle ACD making $AC = 1$ inch and $AD = 2{\cdot}7$. Produce AC to B making $AB = 3$ inches and bisect AB at G. Draw GO vertical to meet AD produced in O. Then COD is a triangle of forces since the reaction of the pivot must pass through O. Thus

$$\frac{\text{Reaction of pivot}}{CO} = \frac{\text{Tension of string}}{DO} = \frac{10 \text{ lbs.}}{OD}.$$

Also, by measurement, $CO = 3{\cdot}32$ ins. and $OD = 1{\cdot}35$ ins. Hence etc.

28. Scale—1 inch = 1 foot. Let DC meet the vertical through B in O; then the third force, the reaction at the hinge, must go through O and AOD is a triangle of forces; so that

$$\frac{\text{Thrust of rod}}{DO} = \frac{\text{Action at hinge}}{AO} = \frac{1 \text{ cwt.}}{AD}.$$

On measurement, $DO = 10{\cdot}82$, $AO = 8{\cdot}49$, and $AD = 8$. Hence etc.

29. Draw GOL a straight line inclined at 50° to the horizon. Make OL one-quarter of GO, so that G is the centre of gravity, O the point of action of the air-thrust, and L the point where the string is tied.

Draw GK vertical and equal to one inch to represent the weight 10 lbs. of the kite. Let GK produced meet the perpendicular to GO through O in N: then LN must be the direction of the string. Draw GM, KM parallel respectively to ON and LN. Then GMK is a triangle of forces so that

$$\frac{\text{Tension of string}}{MK} = \frac{\text{Air-thrust}}{GM} = \frac{10 \text{ lbs.}}{KG}.$$

On measurement, $MK = 2{\cdot}68$ and $MG = 3{\cdot}21$ inches.

EXAMPLES. XII. (Pages 107—111.)

1. [Cf. the figure p. 101, with C for O.] Let G be the middle point of AB, and T be the tension of the string CA; and let R and S be the reactions at A and B, respectively vertical and perpendicular to CB. Since $CA=CB$, the $\angle CAB=$ the $\angle ABC=30°$. Resolving horizontally, we have

$$T=S\sin 60°=\frac{S\sqrt{3}}{2}.$$

Also, taking moments about A, we have

$$W.AG\cos 30°=S.AB\cos 30°,\ \textit{i.e.}\ S=\frac{W}{2};$$

hence $$T=\frac{W}{2}.\frac{\sqrt{3}}{2}=\frac{W\sqrt{3}}{4}.$$

2. [Cf. the figure p. 104, with W for 192, and $\alpha=60°$.] If S be the required force, resolving horizontally, we have $S=R_1$. Also, taking moments about A, we have

$$R_1.AB\sin 60°=W.AG\cos 60°;$$

hence $$R_1=\frac{W}{2}\cot 60°=\frac{W}{2\sqrt{3}}=\frac{W\sqrt{3}}{6}=S.$$

3. The reaction R_1 at B is horizontal, and that at A is vertical. Resolving horizontally, we have $R_1=T\cos\phi$. Also, taking moments about A, we have

$$R_1.AB\sin\theta=W.AC\cos\theta+T.AD\sin\phi;$$

$$\therefore\ T\cos\phi\,(a+b)\sin\theta=W.a\cos\theta+T\,(a+b)\cos\theta\sin\phi,$$

whence $$T=W\frac{a\cos\theta}{(a+b)\sin(\theta-\phi)}.$$

4. [Cf. the figure p. 104.] Let S be the required tension. Resolving horizontally, we have $S=R_1$. Also, taking moments about A, we have

$$R_1.AB\sin\alpha=W.AG\cos\alpha,\ \text{so that}\ S=R_1=\frac{W}{2}\cot\alpha.$$

With the man on the ladder, we have

$$R_1.AB\sin\alpha=W.AG\cos\alpha+\frac{W}{2}.\frac{2}{3}AB\cos\alpha;$$

whence $$R_1=\frac{5W}{6}\cot\alpha=S.$$

5. Let AB be the beam, AO be the horizontal, and OB be the inclined plane. Let R and S be the reactions at A and B, respectively vertical and perpendicular to OB; and let θ be the inclination of the beam to the horizon. Taking moments about B, we have

$$W.a\cos\theta=R.2a\cos\theta,$$

where $2a$ is the length of the beam; *i.e.* $W=2R$, for all values of θ. Also, resolving parallel to the inclined plane, we have

$$P+R\sin\alpha=W\sin\alpha;$$

hence $$P=\sin\alpha\left(W-\frac{W}{2}\right),\ \textit{i.e.}\ 2P=W\sin\alpha.$$

6. Proceed as in Ex. 2, p. 102, with $a=b$, since here the beam is uniform.

7. Let AB be the beam, A being the point where the ground and wall meet; W be its weight acting at its middle point; D be the staple; and T be the tension of the string BD. Since $BD=AD$, the $\angle DAB=$ the $\angle DBA$ ($=\theta$, say). Taking moments about A, to avoid the unknown reaction there, we have

$$T.AB\sin\theta=W.\frac{AB}{2}\sin\theta,\ \textit{i.e.}\ T=\frac{W}{2}.$$

8. The weight of the rod, 2 lbs., acts at G, the middle point of BC. We have $AB=10$ ins., $AC=8$ ins., and $BC=6$ ins.; and since $(10)^2=8^2+6^2$, the angle ACB is a right angle. Let T be the tension of the string AC, and let the angle ABC be θ; then, taking moments about B, to avoid the unknown reaction there, we have

$$T.6=2\times 3\cos\theta,$$

whence $$T=1\times\frac{6}{10}=\frac{3}{5}\text{ lb. wt.}$$

9. Let AB be the rod, A being the upper end, let C be its middle point where its weight W acts; and let BD be the string. Then DA is horizontal, and $BD=DA$. If BD produced meet the line of action of W in the point F, then R the reaction from A acts along AF. Let CF meet DA in E, and let the $\angle DAB=\theta=$ the $\angle DBA$; then the $\angle FDE=2\theta$. Also T the tension of the string $=W$, by hypothesis. Taking moments about A, we have

$$W.AC\sin ACE=T.AB\sin ABF,$$

i.e. $$W\times\frac{1}{2}\cos\theta=W\sin\theta,\ \text{whence}\ \tan\theta=\frac{1}{2}.$$

Also, if the $\angle EAF=\phi$, we have

$$R\cos\phi=T\cos 2\theta,\ \text{and}\ R\sin\phi+T\sin 2\theta=W.$$

Now $$\sin\theta=\frac{1}{\sqrt{5}},\ \text{and}\ \cos\theta=\frac{2}{\sqrt{5}},$$

so that $$\sin 2\theta=2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}}=\frac{4}{5},\ \text{and}\ \cos 2\theta=\cos^2\theta-\sin^2\theta=\frac{3}{5};$$

hence $$R\cos\phi=\frac{3}{5}W\ldots\ldots\ldots\ldots\text{(i)},$$

and $$R\sin\phi=W-\frac{4}{5}W=\frac{1}{5}W\ldots\ldots\ldots\ldots\text{(ii)}.$$

Hence, by squaring and adding (i) and (ii), we have

$$R^2=\frac{10}{25}W^2, \text{ so that } R=\frac{W}{5}\sqrt{10};$$

also, dividing (ii) by (i), $\tan\phi=\frac{1}{3}$.

10. Let AB be the rod, G be its middle point where its weight W acts, A the hinge, and let T be the tension of the string. Taking moments about A, we have

$$T.AB\sin B=W.AG\sin A+\frac{W}{2}.AB\sin A,$$

i.e.
$$T\sin B=\frac{W}{2}\sin A+\frac{W}{2}\sin A=W\sin A;$$

$$\therefore\ T=W\frac{\sin A}{\sin B}=\frac{Wl}{c}.$$

11. W, the weight of the rod, acts at G its middle point. Let R be the action at C perpendicular to AB and let T be the tension of the string. Since AD and CD are equal, the angle ACD = angle $DAC=\theta$, say. Resolving parallel to AB for the equilibrium of the ring, we have

$$2W\sin\theta=T\cos\theta;$$

and for the equilibrium of the whole, taking moments about A, we have

$$W.4a\cos\theta+2W.AC\cos\theta=T.AC\sin\theta.$$

$$\therefore\ 4a\cos\theta=AC\left(\frac{2\sin^2\theta}{\cos\theta}-2\cos\theta\right)=4a\cos\theta\left(-\frac{\cos 2\theta}{\cos\theta}\right).$$

$$\therefore\ \cos 2\theta=-\cos\theta,$$

i.e. $2\theta=\pi-\theta$, whence $\theta=\frac{\pi}{3}$, so that $AC=AD=a$;

also
$$T=2W\tan\theta=2W\sqrt{3}.$$

Again, if X and Y be the horizontal and vertical components of the action at A on the rod, for the equilibrium of the rod and ring, we have

$$X=T\cos\theta=2W\sqrt{3}.\frac{1}{2}=W\sqrt{3},$$

and
$$Y=3W-T\sin\theta=3W-2W\sqrt{3}.\frac{\sqrt{3}}{2}=0;$$

hence the action on the rod at A is a horizontal force equal to $W\sqrt{3}$.

12. Let ACB be the wire, resting at the point C on the horizontal plane; and let O be the centre of the whole circle, so that OC is vertical. Also let the weight P be suspended at A, and the weight Q

at B; then the $\angle AOC=\theta$, and the $\angle BOC=\alpha-\theta$. Let r be the radius of the circle; then, taking moments about O, we have

$$P\,.\,r\sin\theta=Q\,.\,r\sin(\alpha-\theta),$$

i.e. $$P\sin\theta=Q(\sin\alpha\cos\theta-\cos\alpha\sin\theta),$$

i.e. $$P\tan\theta=Q\sin\alpha-Q\cos\alpha\tan\theta,$$

whence $$\tan\theta=\frac{Q\sin\alpha}{P+Q\cos\alpha}.$$

13. Let O be the centre of the bowl, and AB be the rod resting against the wall at A and on the bowl at B; and let D be the point of the bowl in contact with the wall. For equilibrium, the direction of R the reaction of the bowl at B (*i.e.* the radius from B), the direction of the reaction of the wall at A (*i.e.* the horizontal line from A), and the vertical through C, the middle point of the rod where its weight W acts, must meet in a point, E. From B draw a straight line vertically upwards to meet the horizontal diameter of the bowl through DO in F, and the horizontal line from A in G. The $\angle BAD=30°$ $=$the $\angle GBA$; also let the $\angle FBO=\theta=$the $\angle OEC$. Then, resolving vertically, we have

$$R\cos\theta=W\text{..........................}(i),$$

and moments about A give

$$W\,.\,AC\sin 30°=R\,.\,AB\sin(30°-\theta),$$

i.e. $$W=4R\sin(30°-\theta)\text{..........................}(ii).$$

From (i) and (ii), $\cos\theta=4\sin(30°-\theta)=2\cos\theta-2\sqrt{3}\sin\theta$,

whence $$2\sqrt{3}\sin\theta=\cos\theta,\text{ and }\tan\theta=\frac{1}{2\sqrt{3}}.$$

Again, the horizontal projection of $AB=$the sum of the projections of OD and BO; hence $AB\cos 60°=OD+BO\sin\theta$

$$=\frac{a}{2}(1+\sin\theta)=\frac{a}{2}\left(1+\frac{1}{\sqrt{13}}\right);$$

$$\therefore\ AB=a\left(1+\frac{1}{\sqrt{13}}\right).$$

14. Let $ABCD$ be the cylindrical vessel; BDE be the rod, B being the lowest point where the horizontal and vertical reactions are X and Y respectively; D be the point where the rod rests on the upper edge of the vessel, and R be the reaction there perpendicular to BD; and let G be the middle point of BE, where its weight, 6 oz., acts. Then we have

$$BD=\sqrt{BC^2+CD^2}=\sqrt{3^2+4^2}=5\text{ ins., and }BG=GE=4\tfrac{1}{2}\text{ ins.}$$

To find R, moments about B give

$$R\times 5=6\times 4\tfrac{1}{2}\times\cos DBC=6\times\frac{9}{2}\times\frac{3}{5},$$

so that $$R=\frac{81}{25}=3{\cdot}24\text{ oz. wt.}$$

Again, resolving horizontally and vertically, we have $X=R\sin DBC$, and $Y=6-R\cos DBC$; hence the pressure at the base $=\sqrt{X^2+Y^2}$

$$=\sqrt{36-12R\cos DBC+R^2}=\sqrt{36-12\times\frac{81}{25}\times\frac{3}{5}+\left(\frac{81}{25}\right)^2}$$

$$=\sqrt{23\cdot1696}=4\cdot8134 \text{ oz. wt., nearly.}$$

15. The ring assumes a sloping position and is held by the nail at its highest point. Thus, if P and Q be the horizontal reactions between the cylinder and the ring at its highest and lowest points respectively, and S be the vertical reaction at the nail, we have $P=Q$, and $S=W$. Hence the ring is in equilibrium under the action of forces forming two couples, whose moments must be equal; therefore, if θ be the angle the ring makes with the horizontal line through the position of the nail, we have

$$P\,.\,2R\sin\theta=W\,.\,r;$$

but $\qquad 2r=2R\cos\theta$, so that $\sin\theta=\sqrt{1-\dfrac{r^2}{R^2}}$;

$$\therefore\ 2P\sqrt{R^2-r^2}=W\,.\,r,\ \textit{i.e.}\ P=Q=W\,.\,r/2\sqrt{R^2-r^2}.$$

16. If O be the obstacle, and C be the centre of the wheel, the moment of F about O must be slightly greater than the moment of W the other way, the pressure on the ground just vanishing. Hence

$$F\,.\,CN>W\,.\,ON,$$

where CN is vertical and ON horizontal;

but $\qquad CN=r-h,$

and $\qquad ON=\sqrt{CO^2-CN^2}=\sqrt{r^2-(r-h)^2}=\sqrt{2rh-h^2};$

$$\therefore\ F>W\,\frac{\sqrt{2rh-h^2}}{r-h}.$$

17. Let AB be the beam, B being the point in contact with the wall, and let D be the peg. Draw DF, equal to b, perpendicular to the wall. For equilibrium, the directions of the reaction at D perpendicular to AB, the horizontal reaction at B, and the weight of the beam, acting through C the middle point of AB, must meet in a point E. Let $\theta=$ the inclination of the beam to the vertical $=\angle DBF=$ the $\angle DEB=$ the $\angle BCE$. Then we have

$$\sin\theta=\frac{DF}{DB}=\frac{DB}{EB}=\frac{EB}{BC},$$

$$\therefore\ \sin^3\theta=\frac{DF}{DB}\cdot\frac{DB}{EB}\cdot\frac{EB}{BC}=\frac{DF}{BC}=\frac{b}{a},$$

$$\therefore\ \sin\theta=\sqrt[3]{\frac{b}{a}},\ \textit{i.e.}\ \theta=\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$

18. If O be the centre of the disc, T be the tension of the band, R be the horizontal reaction at D, and the $\angle BAD = 2a$, then

$$AB = AD = b, \quad \text{and} \quad \tan a = \frac{a}{b}.$$

The disc being supported by R and the equal tensions T along DA and BA, we have, by resolving vertically,

$$T + T\cos 2a = W, \text{ so that } T = \frac{W}{1+\cos 2a} = \frac{W}{2\cos^2 a},$$

i.e.
$$T = W \div \frac{2b^2}{a^2+b^2} = \frac{W}{2} \cdot \frac{a^2+b^2}{b^2}.$$

Also, resolving horizontally, we have

$$R = T\sin 2a = \frac{W}{2\cos^2 a} \cdot 2\sin a\cos a = W\tan a = \frac{Wa}{b}.$$

19. If P and Q be the pegs, R the action (normal to the rod) at P, T the tension of the string, which is horizontal by symmetry, $2a$ the length of either rod, and if A be the upper end of that which rests on P, then resolving vertically for this rod, we have $W = R\cos\theta$, W being the weight of either rod. Also moments about A give

$$W \cdot a\cos\theta = R \cdot AP, \text{ so that } a\cos^2\theta = AP;$$

but
$$PQ = a + 2 \cdot AP\cos\theta,$$
and therefore
$$2 \cdot AP\cos\theta = a,$$
since
$$PQ = 2a;$$
hence
$$2 \cdot a\cos^2\theta\cos\theta = a, \quad i.e. \quad 2\cos^3\theta = 1.$$

20. Let AB be the rod, G be its middle point where W acts, and let the strings carrying weights w_1 and w_2 be attached at A and B respectively. For equilibrium, the vertical through G and the directions of the strings must meet in a point O. Let θ, a and β be the angles AB, AO and BO make with the vertical respectively. Then, by Lami's Theorem, we have

$$\frac{w_1}{\sin\beta} = \frac{w_2}{\sin a} = \frac{W}{\sin(a+\beta)}.$$

Also the relation $\dfrac{AG}{GO} = \dfrac{BG}{GO}$ gives $\dfrac{\sin a}{\sin(\theta+a)} = \dfrac{\sin\beta}{\sin(\theta-\beta)}$,

i.e.
$$\sin a\sin(\theta-\beta) = \sin\beta\sin(\theta+a).$$

$$\therefore \cot\beta - \cot\theta = \cot a + \cot\theta.$$

$$\therefore \cot\theta = \frac{\cot\beta - \cot a}{2} = \frac{\sin(a-\beta)}{2\sin a\sin\beta}.$$

$$\therefore \cos\theta = \frac{\sin(a-\beta)}{\sqrt{\sin^2(a-\beta) + 4\sin^2 a\sin^2\beta}}.$$

Now $\sin^2(a-\beta) + 4\sin^2 a\sin^2\beta = \sin^2(a-\beta) + [\cos(a-\beta) - \cos(a+\beta)]^2$
$= 1 - 2\cos(a-\beta)\cos(a+\beta) + \cos^2(a+\beta)$
$= 2 - 2(\cos^2 a - \sin^2\beta) - \sin^2(a+\beta) = 2(\sin^2 a + \sin^2\beta) - \sin^2(a+\beta);$

hence $\theta = \cos^{-1} \dfrac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{\sin(\alpha+\beta)\sqrt{2(\sin^2\alpha+\sin^2\beta)-\sin^2(\alpha+\beta)}}$

$$= \cos^{-1} \frac{\sin^2\alpha - \sin^2\beta}{\sin(\alpha+\beta)\sqrt{2(\sin^2\alpha+\sin^2\beta)-\sin^2(\alpha+\beta)}},$$

and the inclination of the rod to the horizon therefore

$$= \sin^{-1} \frac{w_1^2 - w_2^2}{W\sqrt{2(w_1^2+w_2^2)-W^2}}.$$

21. Let AB be the rod, B being the end to which W' is attached. Let C be the middle point of AB where W acts, P be the peg, and x be the length of the string that will slip over. Then $AP = l+x$, and $BP = l-x$. Also, G the centre of the parallel forces W and W' is vertically below P, and AP and BP are equally inclined to the vertical. By Geometry, we have

$$\frac{AG}{GB} = \frac{AP}{BP} = \frac{l+x}{l-x} \qquad \text{(1)},$$

also $$\frac{CG}{GB} = \frac{W'}{W}.$$

$$\therefore \frac{CG}{CB} = \frac{W'}{W+W'}; \text{ but, from (1), } \frac{x}{l} = \frac{AG-GB}{AG+GB} = \frac{2CG}{AB} = \frac{CG}{CB};$$

hence $$\frac{x}{l} = \frac{W'}{W+W'}, \text{ and } x = \frac{lW'}{W+W'}.$$

22. Let D be the middle point of the string, and G be the middle point of the rod AB. The line through G parallel to the vertical wall AC bisects BC, and therefore, by symmetry, passes through D. Let T be the tension of the string, the $\angle CDB = 2\theta$, and the $\angle CAB = \phi$. For the equilibrium of the ring, we have $2T\cos\theta = W$. For the equilibrium of the rod, take moments about A to avoid the unknown reaction there, and we have

$$T \,.\, 2a\sin\left[\left(\frac{\pi}{2}-\phi\right)-\left(\frac{\pi}{2}-\theta\right)\right] = \lambda W \,.\, a\sin\phi;$$

$$\therefore 2T\sin(\theta-\phi) = \lambda \,.\, 2T\cos\theta\sin\phi,$$

i.e. $$\sin\theta\cos\phi = (\lambda+1)\cos\theta\sin\phi \qquad \text{(1)},$$

but $$2l\sin\theta = BC = 2a\sin\phi, \text{ and } 2b = 2a\cos\phi;$$

hence (1) becomes $$a\cos\phi = (\lambda+1)\,l\cos\theta,$$

i.e. $$b = (\lambda+1)\,l\cos\theta = (\lambda+1)\sqrt{l^2 - a^2\sin^2\phi},$$

$$\therefore b^2 = (\lambda+1)^2\left[l^2 - a^2\left(1-\frac{b^2}{a^2}\right)\right] = (\lambda+1)^2(l^2-a^2+b^2);$$

hence $$(\lambda+1)^2(l^2-a^2) = -b^2(\lambda^2+2\lambda),$$

i.e. $$l^2 = a^2 - b^2\frac{\lambda(\lambda+2)}{(\lambda+1)^2}.$$

23. Let W be the weight of the board, acting through G its centre which is vertically under O, T be the tension of the strings, and the $\angle AOG$ be denoted by θ. Resolving vertically, we have $4T\cos\theta=W$;

but $$\cos\theta=\frac{OG}{OD}, \text{ and } OD^2=OG^2+GD^2,$$

i.e. $$OD=\sqrt{b^2+\left(\frac{b}{\sqrt{2}}\right)^2}=\sqrt{\frac{3b^2}{2}}, \text{ so that } \cos\theta=\sqrt{\frac{2}{3}};$$

hence $$T=W\div 4\sqrt{\frac{2}{3}}=\frac{W\sqrt{6}}{8}.$$

24. Let P be the reaction at the upper hinge. There is equilibrium under the action of two couples. Equating moments, we have $P\times 3=100\times 4$, whence $P=133\frac{1}{3}$ lbs. wt. Also, the reaction at the lower hinge

$$=\sqrt{P^2+(100)^2}=100\sqrt{\frac{16}{9}+1}=100\times\frac{5}{3}=166\tfrac{2}{3} \text{ lbs. wt.}$$

25. Let ABC be the triangle; O the centre of the sphere is vertically above I the in-centre of ABC. Let the sphere meet the rods BC, CA, AB (*i.e.* a, b, and c) in the points D, E, and F respectively. The required pressures (R_1, R_2, and R_3, say, on BC, CA, and AB respectively) are normal to the surface of the sphere and, therefore, pass through O, *i.e.* act along DO, EO, and FO; and if DO, EO, and FO make angles α, β, and γ with the vertical, the horizontal components $R_1\sin\alpha$, $R_2\sin\beta$, and $R_3\sin\gamma$ (acting along DI, EI, and FI respectively) are in equilibrium. Now $EIF=180^\circ-A$, $FID=180^\circ-B$ and $EID=180^\circ-C$; hence, by Lami's Theorem, we have

$$\frac{R_1\sin\alpha}{\sin A}=\frac{R_2\sin\beta}{\sin B}=\frac{R_3\sin\gamma}{\sin C};$$

but $\alpha=\beta=\gamma$, since the triangles OID, OIE, and OIF are equal;

$$\therefore \frac{R_1}{\sin A}=\frac{R_2}{\sin B}=\frac{R_3}{\sin C},$$

i.e. $$\frac{R_1}{a}=\frac{R_2}{b}=\frac{R_3}{c}.$$

26. Let P and Q be the pressures on the supports A and B respectively, and let the vertical through C (*i.e.* the line of action of the mass of 18 lbs.) meet AB in D. We have

$$\cos ABC=\frac{(5)^2+(18)^2-(18)^2}{2\times 5\times 18}=\frac{5}{36};$$

and moments about B give $P.AB=18.BD$,

i.e. $$P.18=18\times 5\cos ABC, \text{ i.e. } P=5\times\frac{5}{36}=\frac{25}{36} \text{ lb. wt.};$$

and $$Q=18-P=17\tfrac{11}{36} \text{ lbs. wt.}$$

27. Let ABC be the triangular framework, BC being the longest side, and, therefore, the weight of 63 lbs. being suspended from A. If R and S be the tensions in BA $(=13$ ins.) and CA $(=20$ ins.) respectively, we have $R\sin B+S\sin C=63$, and $R\cos B=S\cos C$. If T be the required tension in BC, we have

$$T=R\cos B=S\cos C;$$

$$\therefore\ T\tan B+T\tan C=63,$$

whence $$T=\frac{63}{\tan B+\tan C}=\frac{63\cos B\cos C}{\sin(B+C)}=\frac{63\cos B\cos C}{\sin A}.$$

Now $$\cos B=\frac{(13)^2+(21)^2-(20)^2}{2\times 13\times 21}=\frac{5}{13},\ \text{so that}\ \sin B=\frac{12}{13};$$

$$\sin C=\frac{13}{20}\sin B=\frac{13}{20}\times\frac{12}{13}=\frac{3}{5},\ \text{so that}\ \cos C=\frac{4}{5};$$

and $$\sin A=\frac{21}{20}\sin B=\frac{21}{20}\times\frac{12}{13}=\frac{63}{65};$$

$$\therefore\ T=63\times\frac{5}{13}\times\frac{4}{5}\div\frac{65}{63}=20\ \text{lbs. wt.}$$

28. Cf. Ex. 4, p. 89, and take that figure, but OC not horizontal, *i.e.* C lower down. Let the vertical from O meet AE in F; then we have the $\angle AOF=\beta$, and the $\angle COF=\alpha$, *i.e.* the $\angle AOC=\alpha+\beta$.

$$\text{Hence the } \angle OAC=\text{the } \angle OCA=90^\circ-\frac{1}{2}(\alpha+\beta),$$

$$\text{and the } \angle OCD=\frac{1}{2}(\alpha+\beta)=\text{the } \angle ADC.$$

Then we have $$AE=AG\cos GAE=AG\cos GDC$$

$$=AG\cos\left[\frac{1}{2}(\alpha+\beta)-\beta\right]=AG\cos\frac{1}{2}(\alpha-\beta);$$

also $$AE=AD\sin ADE=2a\sin\beta;$$

hence $$AG\cos\frac{1}{2}(\alpha-\beta)=2a\sin\beta,$$

$$\textit{i.e. } AB=2AG=4a\sin\beta\sec\frac{1}{2}(\alpha-\beta).$$

Otherwise thus:

$$\frac{AG}{AD}=\frac{\sin ADG}{\sin AGD}=\frac{\sin\beta}{\sin\left[90^\circ+\frac{1}{2}(\alpha+\beta)-\beta\right]}$$

$$=\frac{\sin\beta}{\cos\frac{1}{2}(\alpha-\beta)};\ \therefore\ AB=2AG=4a\sin\beta\sec\frac{1}{2}(\alpha-\beta).$$

EXAMPLES. XIII. (Page 113.)

1. [Take the figure p. 112, with 2, 4, 6, 8 for 1, 9, 5, 3 respectively, and 6 acting along CD.] Let the side of the square be a.
The force 2 is equivalent to a force 2 along OX

together with a couple of moment $2 \cdot \frac{a}{2}$;

the force 4 is equivalent to a force 4 along OY

together with a couple of moment $4 \cdot \frac{a}{2}$;

the force 6 is equivalent to a force -6 along OX

together with a couple of moment $6 \cdot \frac{a}{2}$;

the force 8 is equivalent to a force -8 along OY

together with a couple of moment $8 \cdot \frac{a}{2}$.

Hence the moment of the resultant couple $= 2\frac{a}{2} + 4\frac{a}{2} + 6\frac{a}{2} + 8\frac{a}{2} = 10a$;

the component force along $OX = 2 - 6 = -4$,

and the component force along $OY = 4 - 8 = -4$;

hence the resultant force $= \sqrt{(-4)^2 + (-4)^2} = 4\sqrt{2}$ lbs. wt. along OA.

2. Take AB and AD coinciding with the fixed lines OX and OY. The component force along AB

$$= 3P + 9\sqrt{2}P \cos 45° - 7P = 5P;$$

the component force along AD

$$= 5P - P - 9\sqrt{2}P \sin 45° = -5P;$$

hence the resultant force $= \sqrt{(5P)^2 + (-5P)^2} = 5P\sqrt{2}$, parallel to DB
Also, if a be the side of the square, the moment of the couple

$$= 5P \cdot a + 7P \cdot a - 9\sqrt{2}P \cdot \frac{a}{\sqrt{2}} = 3Pa.$$

3. Take AB and AE coinciding with the fixed lines OX and OY. The component force along AB

$$= 1 + 2\cos 60° - 3\cos 60° - 4 - 5\cos 60° + 6\cos 60° = -3;$$

the component force along AE

$$= 2\sin 60° + 3\sin 60° - 5\sin 60° - 6\sin 60° = -3\sqrt{3};$$

hence the resultant force $= \sqrt{(-3)^2 + (-3\sqrt{3})^2} = 6$ lbs. wt. Also, if the resultant be at an angle θ with AB, we have

$$\tan\theta = \frac{-3\sqrt{3}}{-3} = \sqrt{3}, \text{ i.e. } \theta = 60°,$$

and the resultant is parallel to CB. The moment of the couple = the sum of the moments of the forces about A

$$= 2a \sin 60^\circ + 3 \cdot 2a \sin 60^\circ + 4 \cdot 2a \sin 60^\circ + 5a \sin 60^\circ$$

$$= \frac{21\sqrt{3}a}{2}, \text{ where } a \text{ is the side of the hexagon.}$$

4. If AB to scale represent the force of 10 lbs. wt., applying the couple so that one force of 4 lbs. wt. is at B in the direction BA, and the other force of 4 lbs. wt. is at C, 2 inches distant and parallel to AB, the system becomes 6 lbs. wt. along AB and 4 lbs. wt. through C parallel to AB. Hence, if BC be divided at D in the ratio 4 : 6, *i.e.* so that $BD : CD = 2 : 3$, and DE be drawn equal and parallel to AB, then DE represents the equivalent single force of 10 lbs. wt.

EXAMPLES. XIV. (Pages 117, 118.)

1. Let $ABCD$ be the plate, and A be the point of suspension. Let the directions of W, the weight of the plate, acting through O its centre, and $\frac{W}{2}$ acting at the corner B, meet the horizontal line through A in E and F respectively; and let the angle BAF be θ. Then, taking moments about A, we have

$$W \cdot AE = \frac{W}{2} \cdot AF, \text{ so that } 2AO \cos OAE = AB \cos\theta,$$

i.e.
$$2 \cdot \frac{AB}{\sqrt{2}} \cos(135^\circ - \theta) = AB \cos\theta.$$

$$\therefore \sqrt{2}(\cos 135^\circ \cos\theta + \sin 135^\circ \sin\theta) = \cos\theta.$$

$$\therefore -\cos\theta + \sin\theta = \cos\theta, \text{ whence } \tan\theta = 2, \text{ i.e. } \theta = \tan^{-1} 2.$$

2. If α be the angle of inclination of the rod, x be its length, W be its weight, and A be the point of the base of the cylinder about which it is on the point of turning, then moments about A for the cylinder and the rod together (to avoid their mutual reactions) give

$$W \cdot 2a = W\left(\frac{x}{2}\cos\alpha - 4a\right),$$

so that
$$x \cos\alpha = 12a;$$

but $\tan\alpha = \frac{3}{4}$, so that $\cos\alpha = \frac{4}{5}$; hence $x = 12a \cdot \frac{5}{4} = 15a$.

3. Let $ABCD$ be the cylinder, BC being the base; and let BDE be the rod, B and D being the points in contact with the lower and upper edges of the cylinder respectively. Let W be the weight of the rod, nW be the weight of the cylinder, and θ be the inclination of the

rod to the horizon. Then, when the cylinder is on the point of falling over at C, the moments about C must be equal, and we have

$$W(a\cos\theta - c) = nW.\frac{c}{2},\ \text{so that}\ a\cos\theta = c + \frac{nc}{2};$$

but
$$\cos\theta = \frac{c}{\sqrt{b^2+c^2}};\ \text{hence}\ a = \frac{2+n}{2}\sqrt{b^2+c^2},$$

i.e.
$$2a = (n+2)\sqrt{b^2+c^2}.$$

4. Let $ABCD$ be the top of the table, and suppose the required weight to be placed at the corner A. Let E and F be the middle points of the sides AB and AD respectively, where two adjacent legs meet the top of the table. Then obviously the line EF bisects the distance between A and BD. In the extreme case of equilibrium, the moments about the line joining the feet of the legs at E and F must be equal; hence, since the weight of the table acts through the centre of BD, the greatest weight that can be placed at A is equal to the weight of the table.

5. Let W be the weight of the table acting through its centre O; and let AE and BF be the two legs midway between which the man, of weight W', sits at M. Let OM meet AB in L, and MN be the diameter through M and O. In the extreme case of equilibrium, the moments about the line joining E and F, the feet of the legs, must be equal and we have

$$W'.LM = W.OL;\ \text{also the}\ \angle AOB = 120^\circ.$$

$$\therefore\ OL = OA\cos 60^\circ = \frac{OA}{2},\ \text{and}$$

$$LM = OM - OL = OA(1-\cos 60^\circ) = \frac{OA}{2};$$

$$\therefore\ W' = W.$$

In the second case, M is the point of the table resting on the ground, and N is the highest point. Let the directions of W and W', through O and N, meet the ground in C and D respectively, and H be the middle point of EF; also let θ be the inclination of MN to the horizon. Then, since $W=W'$, moments about EF give

$$CH = HD,\ \textit{i.e.}\ MH - MC = MD - MH.$$

Hence

$$2MH = MC + MD,\ \textit{i.e.}\ 2ML\sec\theta = MO\cos\theta + MN\cos\theta;$$

hence, if r be the radius of the table, and l be the length of the leg, we have $r\sec^2\theta = 3r$, so that $\sec^2\theta = 3$; hence $\tan^2\theta = 2$, and $\tan\theta = \sqrt{2}$.

Also
$$\tan\theta = \frac{LH}{ML} = l \div \frac{r}{2} = \frac{2l}{r};$$

$$\therefore\ \sqrt{2} = \frac{2l}{r},\ \text{and}\ r = l\sqrt{2}.$$

6. Let A, B and C be the points at equal distances where the legs meet the top of the table. The least weight to overturn the table must evidently be hung at the middle point of one of the arcs AB, BC or CA; suppose it hung at M, the middle point of the arc AB. Let W lbs. be the required weight, O be the centre of the table through which its weight, 10 lbs., acts, and let OM meet the chord AB in L. In the extreme case of equilibrium, the moments about the line joining the feet of the legs at A and B must be equal, and we have

$$W \,.\, LM = 10 \,.\, OL,$$

and, as in the first part of the last example, $OL = LM$; hence

$$W = 10 \text{ lbs.}$$

7. Let A, B and C be the tops of the remaining legs, and W be the weight of the table acting through O its centre. By symmetry, the equal weight should be placed on BO, at the point P, say; and taking moments about AOC, the pressure on each leg being $\frac{2W}{3}$, we have

$$\frac{2W}{3} \,.\, BO = W \,.\, OP.$$

$$\therefore\ OP = \frac{2}{3} BO = \frac{1}{3} \text{ diagonal of the square.}$$

8. Let W lbs. be the required weight. In the extreme case of equilibrium, the moments about the line joining the feet of the two legs nearest to W must be equal; hence, if d be the diagonal of the square, we have

$$W \,.\, \frac{d}{4} = 60 \,.\, \frac{d}{4} + 20 \,.\, \frac{3d}{4},$$

whence $$W = 120 \text{ lbs.}$$

9. If θ be the angle which the diameter through A, the point of suspension, makes with the vertical, and r be the radius of the plate, then taking moments about A, we have

$$w \,.\, r \sin\theta = p\,(r - r\sin\theta);$$

whence $$\sin\theta = \frac{p}{p+w}, \text{ i.e. } \theta = \sin^{-1}\frac{p}{p+w}.$$

10. Let O be the centre of the disc, and C be the point on the rim to which W is attached. Then, since there is a weight nW at O, and W at C, we have $(n+1)\,W$ at G on OC, where

$$OG : GC = 1 : n; \text{ i.e. } OG = \frac{OC}{n+1}.$$

When the disc is suspended from A, AG is vertical, and B is the lowest point of the vertical diameter; hence OB is vertical, and therefore parallel to AG. Similarly OA is parallel to BG. Thus $OAGB$ is a parallelogram, and since $OA = OB$ (being radii), it is a

rhombus; therefore its diagonals bisect its angles and meet, at E say, in OC, at right angles to one another; therefore the

$$\angle AOB = 2\angle AOE;$$

and
$$\sec AOE = OA \div \frac{OG}{2} = 2(n+1),$$

since
$$OA = OC,$$

i.e. the
$$\angle AOE = \sec^{-1} 2(n+1);$$

hence the
$$\angle AOB = 2\sec^{-1} 2(n+1).$$

11. O, the centre of gravity of the ring, the point where its weight W acts, is the circumcentre of the triangle ABC. Let P denote the pressure at A; draw AE and OD perpendicular to the side BC of the triangle ABC. Then moments about BC give

$$P \cdot AE = W \cdot OD,$$

i.e.
$$P \cdot c \sin B = W \cdot R \cos A,$$

where
$$R = \frac{a}{2\sin A},$$

i.e.
$$P \cdot c \sin B = W \cdot \frac{a \cos A}{2 \sin A};$$

$$\therefore\ P = \frac{Wa \cos A}{2c \sin B \sin A} = \frac{W \cos A}{2 \sin B \sin C},$$

since
$$c \sin A = a \sin C.$$

Similarly, the pressure at $B = \dfrac{W \cos B}{2 \sin C \sin A}$,

and the pressure at $C = \dfrac{W \cos C}{2 \sin A \sin B}$.

EXAMPLES. XV. (Pages 128, 129.)

Since the centre of gravity of a triangle is on the line joining the middle point of any side to the opposite vertex at one-third of the distance of the vertex from that side, we have, by similar triangles, the perpendicular distance of the centre of gravity from any side is one-third of the perpendicular distance of the opposite vertex from that side.

1. Let AB and AC be the two equal sides of the triangle ABC, and G be its centre of gravity. Draw AD perpendicular to the base BC, bisecting it in D, and GE and GF perpendicular to AC and AB respectively. Then we have

$$GD = \frac{1}{3} AD = \frac{1}{3}\sqrt{AB^2 - BD^2} = \frac{1}{3}\sqrt{5^2 - 3^2} = \frac{4}{3} = 1\tfrac{1}{3} \text{ ft.}$$

Also $$GE=\frac{1}{3}p,$$

where p is the perpendicular distance of B from AC.

Now $p.AC=2$ area of the triangle$=AD.BC$,

so that $$p=\frac{4\times 6}{5}=\frac{24}{5}\text{ ft.};$$

hence $$GE=\frac{8}{5}=1\tfrac{3}{5}\text{ ft.}=GF,\text{ similarly.}$$

2. Let AB (10 ft.), BC (8 ft.), and CA (6 ft.) be the sides of the lamina. Since $(10)^2=6^2+8^2$, the angle ACB is a right angle. Let G be the centre of gravity; draw GD, GE and GF perpendicular to BC, CA and AB respectively. Then we have

$$GD=\frac{1}{3}AC=2\text{ ft.},\quad GE=\frac{1}{3}BC=2\tfrac{2}{3}\text{ ft.},\text{ and }GF=\frac{1}{3}p,$$

where p is the perpendicular distance of C from AB.

Now $p.AB=2$ area of the triangle$=AC.BC$,

so that $$p=\frac{6\times 8}{10}=\frac{24}{5}\text{ ft.};$$

hence $$GF=\frac{8}{5}=1\tfrac{3}{5}\text{ ft.}$$

3. Let ABC be the triangle, and G be its centre of gravity. Draw AD perpendicular to the base BC, bisecting it in D, and join GB and GC. Then we have

$$AD=\sqrt{AB^2-BD^2}=\sqrt{7^2-2^2}=\sqrt{45}=3\sqrt{5}\text{ ins.},$$

and $$GD=\frac{1}{3}AD=\sqrt{5}\text{ ins.};$$

$$\therefore\ GA=\frac{2}{3}AD=2\sqrt{5}\text{ ins.},$$

and $$GB=\sqrt{BD^2+GD^2}=\sqrt{2^2+(\sqrt{5})^2}=3\text{ ins.}=GC.$$

4. If E and F be the middle points of BD and DC respectively; and G_1 and G_2 be the centres of gravity of the triangles ABD and ACD respectively, then

$$AG_1=\frac{2}{3}AE,\text{ and }AG_2=\frac{2}{3}AF;$$

$$\therefore\ AG_1:AG_2=AE:AF;$$

$\therefore$ G_1G_2 is parallel to EF, and in the same proportion to EF;

$$\therefore\ G_1G_2=\frac{2}{3}EF;\text{ but }EF=\frac{1}{2}BC,\text{ so that }G_1G_2=\frac{1}{3}BC.$$

5. The distance of the centre of gravity of a triangle from *any* side is one-third of the distance between that side and the opposite vertex; hence the force is just one-third of the weight of the plate, and will therefore produce the same effect when applied at any vertex.

6. Let W be placed at a distance a from the side BC of the triangular board ABC. The weight of the board gives a pressure $\frac{w}{3}$ at each corner. If P, Q and R be the pressures at A, B and C, respectively, due to the weight W, and AD be perpendicular to BC, then moments about BC give

$$P \, . \, AD = W \, . \, a.$$

$$\therefore \; P = \frac{Wa}{c \sin B};$$

$$\therefore \text{ the total pressure at } A = \frac{w}{3} + \frac{Wa}{c \sin B}.$$

Similarly, the pressures at B and C are

$$\frac{w}{3} + \frac{W\beta}{a \sin C} \quad \text{and} \quad \frac{w}{3} + \frac{W\gamma}{b \sin A}.$$

7. If D be the middle point of the base BC, A and A' be two positions of the vertex, and G and G' be the corresponding positions of the centre of gravity of the triangle, then

$$DG = \frac{1}{3} DA, \quad \text{and} \quad DG' = \frac{1}{3} DA'.$$

Therefore, since
$$\frac{DG}{DA} = \frac{DG'}{DA'},$$
the straight line GG' is parallel to the straight line AA', by Geometry. Hence, since AA' is a fixed line, the centre of gravity moves on a straight line through G any one of its positions.

8. If in any position A of the vertex, the base being BC, G be the centre of gravity of the triangle, D be the middle point of BC, and GE and GF be drawn parallel to AB and AC respectively, to meet BC in E and F, then $DE = \frac{1}{3} DB$; therefore E is a fixed point, and so F. Also the angle EGF is equal to the angle BAC, and is, therefore, constant. Hence the locus of G is a certain circle passing through E and F. If A be restricted to lie on one side only of BC, so also will G, and G therefore moves on an arc of a certain circle.

9. If ABC be the triangle, G be its centre of gravity, and P be any point in BC, then wherever the given weight be placed in the line GP, the centre of gravity, X, of the system will lie within PG, and its greatest distance from G is when the weight is at P, and then X divides PG in a ratio which is fixed, viz. that of the weight of the

triangle to the given weight. Thus it follows that when the given weight is placed within the triangle BGC, X lies within a triangle of which the vertex is G and the base a straight line parallel to BC and terminated by BG and CG. Similarly, if the given weight be placed within ABG or ACG; and in each case the line joining G to the particle is divided in the same ratio; thus the bases of the limiting triangles within BGC and BGA meet on BG; and similarly for AG and CG. Hence X lies within a certain triangle, similar and similarly placed to the given triangle.

10. If A be the angle nearest to D, the point of division of AC, G be the centre of gravity of the plate, and F be the middle point of AB, then

$$AD : DC = 1 : 2 = FG : GC;$$

therefore DG is parallel to AB, and, therefore, the angle $GDC = 60°$; but when suspended from D, DG is a vertical straight line; hence AC is at an angle of $60°$ to the vertical.

11. Let ABC be the lamina, A being the right angle; and let D be the middle point of BC; then, for equilibrium, D must be vertically below A, since the centre of gravity of the lamina is on AD. Hence, if $AC = 3AB$, then $BC = AB\sqrt{10}$, and

$$AD = \frac{1}{2}BC = \frac{AB\sqrt{10}}{2}.$$

Also
$$\sin ADB = \frac{p}{AD},$$

where p is the perpendicular distance of A from BC; now

$$p \,.\, BC = 2 \text{ area of the lamina} = BA \,.\, AC,$$

so that
$$p = \frac{3AB^2}{AB\sqrt{10}} = \frac{3AB}{\sqrt{10}};$$

hence
$$\sin ADB = \frac{3AB}{\sqrt{10}} \div \frac{AB\sqrt{10}}{2} = \frac{3}{5},$$

i.e. the hypotenuse is inclined at an angle $\sin^{-1}\frac{3}{5}$ to the vertical.

12. Let ABC be the lamina, and D be the middle point of AB the longest side. Since $5^2 = 3^2 + 4^2$, the angle ACB is a right angle; hence $CD = AD = DB = \frac{5}{2}$ ins. Also DC is a vertical straight line, since the centre of gravity of the lamina is on DC. Thus we have

$$\cos ADC = \left[\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 3^2\right] \div \left[2 \times \frac{5}{2} \times \frac{5}{2}\right] = \frac{7}{25};$$

i.e. the inclination of AB to the vertical is the angle $\cos^{-1}\frac{7}{25}$, *i.e.* $73°44'$.

EXAMPLES. XVI. (Pages 131, 132.)

1. Let AB ($=12$ ins.) be the rod. Its weight acts at its middle point. If the ounce of lead be fastened at the end A, and X be the required centre of gravity, we have

$$AX=\frac{1\times 0+1\times 6+1\times 8}{1+1+1}=\frac{14}{3}=4\tfrac{2}{3}\text{ ins.}$$

2. Let AB be the bar, and let the given distances be measured from the end A. Then, if X be the point required, we have

$$AX=\frac{3\times 3+3\times 15+6\times 18+3\times 21}{3+3+6+3}=\frac{225}{15}=15\text{ ins.}$$

3. If X be the required centre of gravity, we have

$$AX=\frac{1\times 0+2\times 1+3\times 2+3\times 2+4\times 3+5\times 4}{1+2+3+3+4+5}=\frac{46}{18}=2\tfrac{5}{9}\text{ ft.}$$

4. The centres of gravity of the tubes are at distances of 5 ins., 15 ins., and 25 ins. from A, the end of the 8 oz. tube. Hence, if G be the required centre of gravity, we have

$$AG=\frac{8\times 5+7\times 15+6\times 25}{8+7+6}=\frac{295}{21}=14\tfrac{1}{21}\text{ ins.,}$$

i.e. G is $\dfrac{20}{21}$ in. from the middle of the telescope.

5. The distance from the end where the first particle is placed

$$=\frac{1\times 0+2\times 1+3\times 2+4\times 3+\ldots\ldots+12\times 11}{1+2+3+4+\ldots\ldots+12}$$

$$=\frac{(1+2+3+\ldots\ldots+11)+(1^2+2^2+3^2+\ldots\ldots+11^2)}{1+2+3+4+\ldots\ldots+12}$$

$$=\left(\frac{11\times 12}{2}+\frac{11\times 12\times 23}{6}\right)\div\left(\frac{12\times 13}{2}\right)$$

$$=\frac{66+506}{78}=\frac{22}{3}=7\tfrac{1}{3}\text{ ins.}$$

6. If a be the distance between any two consecutive weights, the distance of the centre of gravity from the first

$$=\frac{1\times 0+4\times a+9\times 2a+16\times 3a}{1+4+9+16}=\frac{7a}{3},$$

and the distance between the extreme weights is $3a$; hence the centre of gravity divides this distance in the ratio of

$$\frac{7a}{3}:3a-\frac{7a}{3},\ \textit{i.e.}\ \frac{7}{3}:\frac{2}{3},\ \textit{i.e.}\ 7:2.$$

7. Let AC be the rod, l be its length, B be its middle point, and D be the point at distance $\frac{l}{3}$ from A. Then, if W and W' be the weights of AB and BC respectively, acting at their middle points, moments about D give

$$W\left(\frac{l}{3}-\frac{l}{4}\right)=W'\left(\frac{2l}{3}-\frac{l}{4}\right).$$

$$\therefore\ W=5W',\ i.e.\ W:W'=5:1.$$

8. Let A be the foot of the inclined plane AB. The distance of G, the centre of gravity, from A along AB

$$=\frac{7\times0+5\times1+4\times2+8\times3}{7+5+4+8}=\frac{37}{24}\text{ ft.}$$

Draw GM perpendicular to the base of the plane; then

$$GM=AG\sin 60^\circ=\frac{37}{24}\times\frac{\sqrt{3}}{2}=1\cdot335\text{ ft.}$$

9. If G be the required centre of gravity, we have

$$AG=\frac{1\times W+2\times2W+3\times3W+\ldots+n\times nW+(n+1)W\times\frac{n}{2}}{(1+2+3+\ldots+n)W+(n+1)W}$$

$$=\left[(1^2+2^2+3^2+\ldots+n^2)+\frac{n(n+1)}{2}\right]$$

$$\div[(1+2+3+\ldots n)+(n+1)]$$

$$=\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]\div\left[\frac{n(n+1)}{2}+(n+1)\right]$$

$$=\frac{2n}{3}\text{ ins.}$$

10. Let AB be the rod, and let its weight W lbs. act at a distance of x feet from A, the end where the mass of 1 lb. is suspended. Then moments about the point at which the rod balances in the two cases give

$$1\times9+W(9-x)=15\times3,$$

and

$$1\times8+W(8-x)=8\times4,$$

i.e.

$$W(9-x)=36\ldots\ldots\ldots\ldots\text{(i)},$$

and

$$W(8-x)=24\ldots\ldots\ldots\ldots\text{(ii)}.$$

Hence, by division,

$$\frac{9-x}{8-x}=\frac{3}{2},\ \text{whence } x=6\text{ ft.};$$

hence the centre of gravity is at the middle point of the rod. Also, from (i) or (ii), $W=12$ lbs.

EXAMPLES. XVII. (Pages 137, 138.)

1. As on p. 135, let the particles of 1, 2, 3 and 4 lbs. wt. be placed at the angular points O, A, B and C respectively. Let a be the side of the square, D be its centre, and G be the required centre of gravity. Since $1+4=3+2$, the centre of gravity is on MN, the line bisecting OA and BC. Also

$$MG=\bar{y}=\frac{1\times 0+2\times 0+3\times a+4\times a}{1+2+3+4}=\frac{7a}{10};$$

but $$MD=\frac{a}{2};\text{ hence } GD=\frac{7a}{10}-\frac{a}{2}=\frac{a}{5}.$$

2. Let the two fixed lines from which the distances are measured be AB and AD. Let G be the required centre of gravity, and draw GM perpendicular to AB. Then, if a be the side of the square, we have

$$MG=\bar{y}=\frac{2\times 0+1\times 0+2\times a+7\times a}{2+1+2+7}=\frac{3a}{4};$$

$$AM=\bar{x}=\frac{2\times 0+1\times a+2\times a+7\times 0}{2+1+2+7}=\frac{a}{4}.$$

3. Let the two fixed lines from which the distances are measured be AB and AD. Let G be the centre of gravity, and draw GM perpendicular to AB. Then

$$MG=\bar{y}=\frac{5\times 0+6\times 0+9\times 27+7\times 27}{5+6+9+7}=16\text{ ins.};$$

$$AM=\bar{x}=\frac{5\times 0+6\times 27+9\times 27+7\times 0}{5+6+9+7}=15\text{ ins.}$$

The force must be applied at G.

4. The distance of the centre of gravity from the first edge

$$=\frac{1\times 2+2\times 4+3\times 6+4\times 8+5\times 10}{1+2+3+4+5}=\frac{110}{15}=7\tfrac{1}{3}\text{ ins.}$$

The distance from the other edge

$$=\frac{1\times 3+2\times 5+3\times 7+4\times 9+5\times 11}{1+2+3+4+5}=\frac{125}{15}=8\tfrac{1}{3}\text{ ins.}$$

5. Let ABC be the triangle, and let the weights proportional to 1, 2 and 3 (W, $2W$ and $3W$, say) be placed at A, B and C respectively. Let the two fixed lines from which the distances are measured be AX, coinciding with AC, and AY, a perpendicular to AX through the point A. Then, if G be the centre of gravity, and GM be drawn perpendicular to AX, we have

$$AM=\frac{W\times 0+2W\times\frac{a}{2}+3W\times a}{W+2W+3W}=\frac{2a}{3},$$

and $$MG=\frac{W\times 0+2W\times\frac{a\sqrt{3}}{2}+3W\times 0}{W+2W+3W}=\frac{a\sqrt{3}}{6}.$$
hence the required distance $=AG$
$$=\sqrt{AM^2+MG^2}=\sqrt{\frac{4a^2}{9}+\frac{3a^2}{36}}=\frac{a}{6}\sqrt{19}.$$

Again, if the weights proportional to 11, 13 and 6 ($11W$, $13W$ and $6W$, say) be placed at A, B and C respectively, we have
$$AM=\frac{11W\times 0+13W\times\frac{a}{2}+6W\times a}{11W+13W+6W}=\frac{5a}{12},$$
and $$MG=\frac{11W\times 0+13W\times\frac{a\sqrt{3}}{2}+6W\times 0}{11W+13W+6W}=\frac{13a\sqrt{3}}{60};$$
$$\therefore\ AG=\sqrt{\frac{25a^2}{144}+\frac{169a^2}{1200}}=\frac{a}{30}\sqrt{283}.$$

6. Let the two fixed lines from which the distances are measured be BX, coinciding with BC, and BY, a perpendicular to BX through the point B. Let D, E and F be the middle points of BC, CA and AB respectively, G be the centre of gravity of the system, and $5W$, W, $3W$, $2W$, $4W$ and $6W$ be the given weights respectively. Draw FH, GM, AD and EK perpendicular to BC. Then we have
$$AB=BC=CA=24 \text{ ins.},$$
$$BH=HD=DK=KC=6 \text{ ins.},$$
$$AD=12\sqrt{3} \text{ ins., and } FH=EK=6\sqrt{3} \text{ ins.}$$
Hence
$$BM=\frac{2W\times 12+3W\times 24+4W\times 18+5W\times 12+6W\times 6}{W+2W+3W+4W+5W+6W}=\frac{88}{7}\text{ ins.},$$
and $$MG=\frac{4W\times 6\sqrt{3}+5W\times 12\sqrt{3}+6W\times 6\sqrt{3}}{W+2W+3W+4W+5W+6W}=\frac{40\sqrt{3}}{7}\text{ ins.};$$
hence the required distance $=BG=\sqrt{BM^2+MG^2}$
$$=\sqrt{256}=16 \text{ ins.}$$

7. If ABC be the lamina, its centre of gravity G may be found by bisecting BC at D and dividing AD in G so that $AG=2GD$. Now 1 oz. at B and 1 oz. at C may be replaced by 2 oz. at D; hence the centre of gravity of these and 1 oz. at A is also at G. Also the masses of 1 oz. at the middle points of the sides may each be replaced by masses of $\frac{1}{2}$ oz. at the ends of that side; they are, therefore, equivalent to 1 oz. at each angular point; hence their centre of gravity is also at G. Thus the centre of gravity of the six masses is at G, the centre of gravity of the lamina.

8. Let G be the centre of gravity, and draw GM perpendicular to AC. Then, if $2W$, $3W$ and $4W$ be the weights, we have

$$AM=\frac{2W\times 0+3W\times 15+4W\times 0}{2W+3W+4W}=\frac{45}{9}=5 \text{ ins.};$$

$$GM=\frac{2W\times 0+3W\times 0+4W\times 12}{2W+3W+4W}=\frac{48}{9}=\frac{16}{3}=5\tfrac{1}{3} \text{ ins.};$$

$$\therefore\ GB^2=\left(12-\frac{16}{3}\right)^2+5^2=\frac{625}{9}, \text{ so that } GB=\frac{25}{3}=8\tfrac{1}{3} \text{ ins.};$$

also $$GC^2=(15-5)^2+\left(\frac{16}{3}\right)^2=\frac{1156}{9}, \text{ so that } GC=\frac{34}{3}=11\tfrac{1}{3} \text{ ins.}$$

9. Let the masses 4, 1 and 1 lbs. be placed at A, B and C respectively, the angular points of the triangle ABC. If D be the middle point of BC, and G be the centre of gravity of the triangle ABC, then G is in AD and $AG=\frac{2}{3}AD$. Also 1 lb. at B and 1 lb. at C are equivalent to 2 lbs. at D; hence G', the centre of gravity of the three particles, is on AD, and such that

$$AG' : G'D=2 : 4=1 : 2;$$

and therefore $$AG' : AG=1 : 3, \text{ i.e. } AG'=\frac{1}{3}AD.$$

Thus $$AG'=\frac{1}{2}AG, \text{ i.e. } G' \text{ bisects } AG.$$

10. One mass being at A, and the centre of inertia of the three at G, the centre of inertia of the other two must be in AG produced; it must also be on BC, and therefore is at D the middle point of BC. Hence the masses at B and C are equal (m and m say); and, since $GA=GD$, if m be the mass at A, clearly $m'=2m$, and the ratios are $2:1:1$

11. Divide BC in D, so that $BD : DC=4:3$; the centre of gravity of masses 3 lbs. and 4 lbs. is at D. Divide AD in G, so that $AG : GD=7:2$; the centre of gravity of the three masses is at G. By Art. 42, the resultant of forces represented by $3GB$ and $4GC$ is $7GD$; but $7GD=2AG$, hence forces represented by $2GA$, $3GB$ and $4GC$ are in equilibrium.

12. If the distances of A and B from BC and AC be a and b respectively, and h and k be the corresponding distances of the required centre of gravity, we have

$$h=\frac{2\times a+3\times 0+5\times 0+3\times\frac{a}{3}}{2+3+5+3}=\frac{3a}{13},$$

and $$k=\frac{2\times 0+3\times b+5\times 0+3\times\frac{b}{3}}{2+3+5+3}=\frac{4b}{13}.$$

13. 2 lbs. at A and 2 lbs. at B are equivalent to 4 lbs. at D, the middle point of AB; therefore we have 11 lbs. at C, 3 lbs. at G and 4 lbs. at D. Hence, if H be the centre of gravity of the system,

$$CH=\frac{11\times 0+3\times CG+4\times CD}{11+3+4}=\frac{3CG+4\times\frac{3}{2}CG}{18}=\frac{9CG}{18}=\frac{1}{2}CG.$$

14. Let $ABCDEF$ be the hexagon, O be its centre, and a be its side; and let the masses be placed at the angular corners A, B, C, D, E and F respectively. G, the required centre of gravity, is obviously on BOE, and we have

$$BG=\frac{3\times 0+4\times\frac{a}{2}+12\times\frac{3a}{2}+9\times 2a}{3+4+12+9}=\frac{19}{14}a=a+\frac{5}{14}a;$$

hence $\quad OG=\frac{5}{14}a$, and $GE=\frac{9}{14}a$, *i.e.* $OG:GE=5:9$.

15. Since $5+4=7+2$, the centre of gravity is in the line joining the points where the weights proportional to 3 and 6 are placed. Similarly, since $7+3=6+4$, the centre of gravity is in the line joining the points where the weights proportional to 2 and 5 are placed. Hence the centre of gravity is at the centre of the hexagon.

16. Proceed as in the last example.

17. If the hexagon be represented by $ABCDEF$, and a be its side, O the point of intersection of the diagonals is the centre of the circumscribing circle, the radius of which $OA=BC=a$. Let the weights be placed at A, B, C, D, E and F respectively; then, since $6+1=4+3$, G the centre of gravity of the system is obviously on BE; and we have

$$BG=\frac{3\times\frac{a}{2}+4\times\frac{3a}{2}+5\times 2a+6\times\frac{3a}{2}+1\times\frac{a}{2}}{2+3+4+5+6+1}=\frac{27}{21}a$$

$$=a+\frac{2}{7}a,\ i.e.\ OG=\frac{2}{7}a.$$

18. Let $ABCD$ be the square, a be its side and O be its centre. Since $7+1=5+3$, G, the required point, is on the line bisecting AB and CD, and

$$OG=\frac{12\times\frac{a}{2}-4\times\frac{a}{2}}{12+4}=\frac{4a}{16}=\frac{a}{4}.$$

19. The forces at the points of bisection of the several sides are equivalent to their halves at the ends of those sides; thus they are equivalent to parallel forces proportional to 4 at A, 4 at B, 6 at B, 6 at C, 8 at C, 8 at D, 2 at D and 2 at A; *i.e.* to 6 at A, 10 at B, 14 at C and 10 at D. Q.E.D.

20. If G be the required centre, we have

$$AG=\frac{P\times 1+2P\times 2+3P\times 3+4P\times 4+5P\times 5+6P\times 6}{P+2P+3P+4P+5P+6P}$$

$$=\frac{1^2+2^2+3^2+4^2+5^2+6^2}{1+2+3+4+5+6}=\left[\frac{6\times 7\times 13}{6}\right]\div\left[\frac{6\times 7}{2}\right]$$

$$=\frac{13}{3}=4\tfrac{1}{3}\text{ ins.}$$

21. The magnitude of the resultant $=P+Q+R$. Divide BC in D so that $BD:DC=R:Q=c:b=BA:AC$; then the resultant of Q and R acts at D; and the resultant of P, Q and R acts at some point in AD. Also, by Euc. VI. 3, AD bisects the angle BAC. Similarly, the resultant of P, Q and R acts at some point in CE which bisects the angle ACB; hence it must act at the intersection of these bisectors, *i.e.* at the centre of the circle inscribed in ABC.

EXAMPLES. XVIII. (Pages 141—143.)

1. Let A be the joint, and B be the middle point of the longer portion. The weights of the two parts (acting at A and B) are proportional to their lengths; let them be denoted by $5W$ and $7W$. Hence, if G, on AB, be the required centre of gravity,

$$AG=\frac{5W\times 0+7W\times\frac{7}{2}}{5W+7W}=\frac{49}{24}=2\tfrac{1}{24}\text{ ins.}$$

2. Let A be the middle point of the lower end of the figure. The weights of the pieces are proportional to their areas, *i.e.* to 6×2 and $8\times 2\frac{1}{2}$; let them be denoted by $12W$ and $20W$; they act at C and B, the middle points of the pieces respectively, where

$$AC=(8+1)\text{ ins.}=9\text{ ins., and }AB=4\text{ ins.}$$

Hence, if G be the required centre of gravity,

$$AG=\frac{12W\times 9+20W\times 4}{12W+20W}=\frac{47}{8}=5\tfrac{7}{8}\text{ ins.}$$

3. Let AC (the shorter) and CB be the two portions of the beam AB; their weights, $3W$ and W, act at D and E, their middle points, respectively. If $AC=3a$, and $CB=5a$, then $AD=\frac{3a}{2}$, and

$$AE=3a+\frac{5a}{2}=\frac{11a}{2}.$$

Hence, if G be the required centre of gravity,

$$AG=\frac{3W\times\frac{3a}{2}+W\times\frac{11a}{2}}{3W+W}=\frac{5a}{2};$$

also
$$GB=8a-\frac{5a}{2}=\frac{11a}{2};$$

therefore G divides AB in the ratio of $5:11$.

4. Let $BCDE$ be the rectangle and ABC be the equilateral triangle, so that BC is the common side. Bisect ED in O, and join AO; the required centre of gravity is clearly in AO. Let G and H be the centres of gravity of the triangle and the rectangle respectively; also let $BC=ED=2a$, so that $BE=CD=a$. The weights of the triangle and the rectangle are proportional to their areas, *i.e.* to $a^2\sqrt{3}$ and $2a^2$ respectively, and act at G and H, where

$$AG=\frac{2}{3}\cdot\frac{2a\sqrt{3}}{2}=\frac{2a}{\sqrt{3}},$$

and
$$AH=\frac{2a\sqrt{3}}{2}+\frac{a}{2}=\frac{a}{2}(1+2\sqrt{3}).$$

Hence, F being the required centre of gravity, we have

$$AF=\frac{a^2\sqrt{3}\times\frac{2a}{\sqrt{3}}+2a^2\times\frac{a}{2}(1+2\sqrt{3})}{a^2\sqrt{3}+2a^2}=a\sqrt{3},$$

i.e. F is at the middle point of BC.

5. Let E be the vertex of the triangle. Bisect AD in F, and join EF. The weights of the triangle and the square are proportional to their areas, *i.e.* to 36 and 144, and act at K and H, their centres of gravity respectively, where

$$FK=\left(12+\frac{1}{3}\times 6\right)\text{ ins.}=14\text{ ins., and } FH=6\text{ ins.}$$

Hence, G being the required centre of gravity, we have

$$FG=\frac{144\times 6+36\times 14}{144+36}=\frac{38}{5}=7\tfrac{3}{5}\text{ ins.}$$

6. Let ABC be the isosceles triangle, A be the right angle, and D be the centre of the hypotenuse. The required centre of gravity is on AD, by symmetry. If

$$AB=AC=a,\text{ then } AD=\frac{a}{\sqrt{2}}.$$

The area of the triangle $ABC=\frac{a^2}{2}$, that of each of the squares on AB

and AC is a^2, and that of the square on BC is $2a^2$. Hence, G being the required centre of gravity, we have

$$AG=\frac{(a^2+a^2)\times 0+\frac{a^2}{2}\times\frac{2}{3}\times\frac{a}{\sqrt{2}}+2a^2\left(\frac{a}{\sqrt{2}}+\frac{a\sqrt{2}}{2}\right)}{a^2+a^2+\frac{a^2}{2}+2a^2}=\frac{26}{27}\cdot\frac{a}{\sqrt{2}};$$

$$\therefore\ DG=\frac{1}{27}\cdot\frac{a}{\sqrt{2}},\ i.e.\ DG:GA=1:26.$$

7. If G_1 and G_2 be the centres, and, therefore, the centres of gravity, of the larger and smaller spheres respectively, and G be the required centre of gravity, we have (since the diameters are as $2:1$, and therefore the volumes as $8:1$)

$$GG_1:GG_2=1:8,$$

i.e.
$$GG_1=\frac{1}{9}G_1G_2=\frac{1}{9}\times 9=1 \text{ inch.}$$

8. Let $ABCD$ be the parallelogram, O the point of intersection of the diagonals, and let the triangle AOB be removed. The centre of gravity of the whole parallelogram is at O. Draw HOK bisecting AB and CD in H and K respectively; G, the required centre of gravity, evidently lies on OK, and if F be the centre of gravity of the triangle AOB, we have

$$GO:OF=\text{wt. at } F:\text{wt. at } G=W:3W=1:3,$$

where W is the weight of each portion; hence

$$OG=\frac{1}{3}OF=\frac{1}{3}\times\frac{2}{3}\times OH=\frac{1}{9}HK.$$

9. Let $ABCD$ be the parallelogram, O the point of intersection of the diagonals, H and K the middle points of AB and AD respectively, and let the part $AHOK$ be removed. Then, if W be the weight of each part, we have $2W$ at O and W at F, the middle point of OC. Hence the required centre of gravity is at G (on OC) where

$$OG:GF=1:2;$$

$$\therefore\ OG=\frac{1}{3}OF=\frac{1}{6}OC=\frac{1}{12}AC.$$

10. Let $ABCD$ be the square, and O be the point of intersection of its diagonals so that O is the centre of gravity of the square. Join the middle points of AB and AD, and let the triangle thus formed be cut off. Let H be the centre of gravity of this triangle, and G be the required centre of gravity; G will be on OC. The portion cut off is one-eighth of the square $ABCD$. Hence we have

$$GO:OH=\text{wt. at } H:\text{wt. at } G=1:7;$$

$$\therefore\ GO=\frac{1}{7}OH=\frac{1}{7}(AO-AH)=\frac{1}{7}\left(AO-\frac{1}{3}AO\right)=\frac{2}{21}AO=\frac{1}{21}AC.$$

11. Proceed as in Ex. 2, Page 141. Here

$$AB_1 = \frac{1}{3}AB, \text{ and } W_2 = 8W_1.$$

12. $\triangle OBC = \frac{1}{3}\triangle ABC$. If AO meet BC in D, H be the centre of gravity of the triangle OBC, and G be the required centre of gravity, we have

$$GO : OH = \text{wt. at } H : \text{wt. at } G = 1 : 2;$$

hence
$$GO = \frac{1}{2}OH = \frac{1}{2}\times\frac{2}{3}OD = \frac{1}{3}OD = \frac{1}{9}AD$$

$$= \frac{1}{9}\times 6\times\frac{\sqrt{3}}{2} = \frac{1}{3}\sqrt{3} \text{ ins.}$$

13. Let w be the weight of each triangle ARQ, BPR and CQP. Resolve their weights into weights $\frac{w}{3}$ acting at their vertices. Then the system of these three triangles is equivalent to a system of weights $\frac{2w}{3}$ at each corner of the triangle PQR with a system $\frac{w}{3}$ at each point A, B and C. Considering the weight of the triangle PQR, w' say, and resolving in the same way, we have altogether $\frac{2w+w'}{3}$ at each point P, Q and R, and $\frac{w}{3}$ at each point A, B and C. But as we have now taken the whole triangle ABC, the centre of inertia of the whole system must be at G, the centre of inertia of the triangle ABC. Also the centre of inertia of the system $\frac{w}{3}$ is known to be G; hence the centre of inertia of the system $\frac{2w+w'}{3}$ must also be at G.

Since the centre of gravity of equal particles at P, Q, and R is at G it follows that the centre of gravity of the lamina PQR is at G.

14. Let AG meet BC in D, H be the centre of gravity of the triangle GBC, and K be the required centre of gravity. Since

$$BC = a, \text{ therefore } DA = \frac{a}{2},$$

and since $\triangle GBC = \frac{1}{3}\triangle ABC$, the remainder $= \frac{2}{3}\triangle ABC$.

Also
$$\triangle ABC = BD \times DA = \frac{a^2}{4}.$$

Hence we have

$$DG = \frac{\triangle BGC \times DH + \text{remainder} \times DK}{\triangle ABC}$$

$$= \frac{1}{3} DH + \frac{2}{3} DK;$$

i.e. $$\frac{a}{6} = \frac{1}{3} \times \frac{a}{18} + \frac{2}{3} DK, \text{ whence } DK = \frac{2a}{9}.$$

$$\therefore\ AK = DA - DK = \frac{a}{2} - \frac{2a}{9} = \frac{5a}{18}.$$

15. Produce AA' to cut BC in D; bisect AA' at E; join EB and EC, and take

$$EG = \frac{1}{3} EB, \text{ and } EG' = \frac{1}{3} EC.$$

The centre of gravity of ABA' is at G, and of ACA' at G'; hence GG' passes through A' by hypothesis, and is parallel to BC;

$$\therefore\ EA' = \frac{1}{3} ED,$$

$$\therefore\ EA' = \frac{1}{2} A'D, \text{ so that } AA' = A'D.$$

Again, $$GA' : A'G' = \triangle A'CA : \triangle ABA'$$
$$= CD : DB;$$

but $$GA' : A'G' = BD : DC.$$
$$\therefore\ BD = DC.$$

Hence A' bisects AD, where D is the middle point of BC.

16. Let ED and HF be lines parallel to the sides AC and AB, respectively, of the triangle ABC, the areas BED and FHC being each $\frac{1}{m}$th of that of ABC. Bisect BC in K, and join AK. Bisect BD in M and FC in N, and join EM and HN. Let G, P and Q be the centres of gravity respectively of ABC, BED and FHC, and G', which will be on AK, be the centre of gravity of the remainder $AEDFH$. Join PQ, meeting AK in R; R is the centre of gravity of the pair of triangles BED and FHC. Then we have

$$G'G : GR = \text{wt. at } R : \text{wt. at } G' = \frac{2}{m} : 1 - \frac{2}{m} = 2 : m - 2 \text{.........(i)}.$$

Now since $$\triangle BED = \frac{1}{m} \triangle ABC = \triangle FHC,$$

we have, by Euc. VI. 19,

$$BE = \frac{1}{\sqrt{m}} BA = FH.$$

So $$BD = FC = \frac{1}{\sqrt{m}} BC, \text{ and } EM = \frac{1}{\sqrt{m}} AK = HN.$$

$$\therefore\ MP=NQ=KR=\frac{1}{3\sqrt{m}}AK.$$

$$\because\ GR=\left(\frac{1}{3}-\frac{1}{3\sqrt{m}}\right)AK=\frac{\sqrt{m}-1}{3\sqrt{m}}AK;$$

hence, from (i),

$$G'G=\frac{2}{m-2}GR=\frac{2(\sqrt{m}-1)}{3\sqrt{m}(m-2)}AK=\frac{\sqrt{m}-1}{m\sqrt{m}-2\sqrt{m}}AG;$$

$$\therefore\ AG':GG'=(AG-GG'):GG'=m\sqrt{m}-2\sqrt{m}-(\sqrt{m}-1):\sqrt{m}-1$$
$$=m\sqrt{m}-3\sqrt{m}+1:\sqrt{m}-1.$$

17. Let $ABCD$ be the square, the side AB being the base of the triangle to be cut out. Let G be the point of intersection of the diagonals of the square, *i.e.* its centre of gravity; and H be the centre of gravity of the triangle. O the vertex of the triangle will be on the line through G bisecting AB and CD; let this line meet AB in E. Then, if $AB=a$, and $EO=x$, the area of the square $=a^2$, and the area of the triangle $=\frac{1}{2}ax$. Hence we have

$$EG=\frac{\triangle AOB\times EH+\text{remainder}\times EO}{\text{sq. } ABCD};$$

i.e.
$$\frac{a}{2}=\frac{\frac{1}{2}ax\times\frac{1}{3}x+\left(a^2-\frac{1}{2}ax\right)\times x}{a^2},$$

whence $\quad 2x^2-6ax+3a^2=0,$ and $x=\frac{3\pm\sqrt{3}}{2}.a.$

Now x must be less than a, so that

$$x=\frac{3-\sqrt{3}}{2}a.$$

18. The area of the plate $=100$ sq. ins., and of the hole $=3$ sq. ins. If O be the centre of the square and H be that of the hole, then G, the required centre of gravity, is on HO produced, so that

$$OG:OH=\text{wt. at } H:\text{wt. at } G=3:97$$

$$\therefore\ OG=\frac{3}{97}\times 2\tfrac{1}{2}=\frac{15}{194}\text{ in.}$$

19. The area of the hole : the area of the plate $=1:3^2=1:9$. If H be the required position of the centre of the hole, C be the centre of the disc, and G be the centre of gravity of the remainder, we have

$$CG=2\text{ ins.},$$

and $\quad CH:CG=\text{wt. at } G:\text{wt. at } H=8:1;$

hence $\quad CH=8CG=16$ ins.

20. The volumes of the spheres are as $a^3 : b^3$. Let A be the centre of the larger sphere, radius a, and B be the centre of the sphere, radius b. Then, if G be the required centre of gravity, we have

$$GA : AB = \text{wt. at } B : \text{wt. at } G = b^3 : a^3 - b^3;$$

hence
$$AG = \frac{(a-b)\,b^3}{a^3-b^3} = \frac{b^3}{a^2+ab+b^2}.$$

21. Let A be the vertex of the cone, h be its height, K be the centre of gravity of the smaller cone, and H be that of the frustum. The volume of the smaller cone is equal to one-eighth of the volume of the whole cone; hence we have

$$AH = \frac{\frac{W}{8} \times AK + \frac{7W}{8} \times AH}{W},$$

i.e.
$$\frac{3}{4}h = \frac{1}{8} \cdot \frac{3h}{8} + \frac{7}{8}AH, \text{ whence } AH = \frac{45}{56}h.$$

22. Let A be the vertex of the whole cone, and B be the centre of the base. The volume of the whole cone : the volume of the smaller cone $= 4096 : 343 = (16)^3 : 7^3$; therefore their heights are as $16 : 7$; *i.e.* the heights are 64 ins. and 28 ins. Let G, G_1, and G_2 be the centres of gravity of the whole cone, the smaller cone, and the truncated portion respectively; then we have

$$BG = \frac{(8192-686)\,BG_2 + 686\,BG_1}{8192},$$

i.e.
$$16 \times 4096 = 3753\,BG_2 + 343\left(64 - \frac{3}{4} \times 28\right);$$

$$\therefore BG_2 = \frac{65536 - 14749}{3753} = 13{\cdot}532 \text{ ins.}$$

23. Let A and B be the vertices of the original cone and the hollow, and G and G_1 be their centres of gravity, respectively. If O be the centre of the base, the volumes are as $AO : BO$, as $h : x$ say.

Then we have $OG = \dfrac{x \times OG_1 + (h-x)\,OB}{x + (h-x)}$, *i.e.* $\dfrac{h}{4} \times h = x \times \dfrac{x}{4} + (h-x)x$,

i.e. $\dfrac{h^2-x^2}{4} = x(h-x)$, *i.e.* $h + x = 4x$, and therefore $x = \dfrac{h}{3}$.

24. Let O, O' be the centres of the earth and moon, and G their common centre of gravity, so that

$OG : GO' :: \text{mass of the moon} : \text{mass of the earth} :: {\cdot}013 : 1.$

$\therefore OG = {\cdot}013\,(OO' - OG)$, so that

$$OG = \tfrac{13}{1013} \cdot OO' = \tfrac{13}{1013} \times 60 \times 4000 \text{ miles} = \text{etc.}$$

EXAMPLES. XIX. (Pages 145—148.)

1. Let ABC be the top of the table, the legs meeting it at the points A, B, and C; and let D, E, and F be the middle points of the sides BC, CA, and AB, where the weights of 6 lbs., 8 lbs., and 10 lbs. are placed respectively. The 6 lbs. at D is equivalent to 3 lbs. at B and 3 lbs. at C; the 8 lbs. at E is equivalent to 4 lbs. at C and 4 lbs. at A; and the 10 lbs. at F is equivalent to 5 lbs. at A and 5 lbs. at B. Hence the pressures at A, B, and C are increased by 9 lbs. wt., 8 lbs. wt., and 7 lbs wt. respectively.

2. The distance from AB of C or D

$$=\sqrt{5^2-\left[\frac{1}{2}(18-12)\right]^2}=\sqrt{5^2-3^2}=4 \text{ ins.};$$

hence the required distance

$$=\frac{12\times 4+2\times 5\times \frac{4}{2}+18\times 0}{12+2\times 5+18} \text{ ins.}=1\tfrac{7}{10} \text{ in.}$$

3. The $\angle ABC=120^\circ$, hence the $\angle BCA=30^\circ$, and therefore AC is perpendicular to CD. Hence the distance of the centre of gravity of CD from $AC=\frac{1}{2}CD$, and the distance from AC of the centre of gravity of AB or $BC=\frac{1}{2}.BA\sin BAC=\frac{1}{4}BA$. Hence the distance from AC of the centre of gravity of the three rods

$$=\frac{2.\frac{1}{4}BA-\frac{1}{2}CD}{2+1}=0;$$

thus the rods when suspended from A will hang so that AC is vertical, and, therefore, CD must be horizontal.

4. The weights of the parts AB and BC are proportional to their lengths, and act at their middle points. Their moments about B must be equal and opposite; hence we have

$$W.BC.\frac{BC}{2}\cos 45^\circ=W.AB.\frac{AB}{2};$$

$$\therefore \frac{BC^2}{2\sqrt{2}}=\frac{AB^2}{2}, \text{ so that } \frac{BC^2}{AB^2}=\frac{\sqrt{2}}{1},$$

i.e. $$BC : AB=\sqrt[4]{2} : 1.$$

5. If A and B be the two ends of the wire, C be the centre of AB, G be the centre of gravity of the wire, and O be the centre of gravity of the complete hexagon, then GOC must be a straight line, by symmetry perpendicular to AB, and $5 \, . \, GO = 1 \, . \, OC$.

$$\therefore \; GC = \frac{6}{5} OC = \frac{6}{5} a \sin 60^\circ = \frac{3\sqrt{3}}{5} a.$$

$$\therefore \; GA^2 = \frac{a^2}{4} + \frac{27a^2}{25} = \frac{25+108}{100} a^2,$$

and $$GA = \frac{a}{10} \sqrt{133} = GB.$$

6. Taking a figure and notation, as in Art. 118, we have

$$CD = 2AB,$$

and therefore $$EG_2 : G_2F = 2DC + AB : 2AB + DC$$
$$= 5 : 4.$$

7. As in Examples XVIII., 15, if CA meet BD in the point E we have $CA = AE$, so that $CE = 2AE$; hence, by similar triangles, we obtain the required result.

8. We have to shew that the centre of gravity of the whole system is in AC. By Art. 104 the weight of a triangle may be replaced by three particles each equal to one-third of its weight placed at its angular points.

Hence we have to shew that the moment about AC of the weight of a particle placed at B equal to the weight of ABC is the same as that of a particle at D equal to one-third of the weight of the triangle ADC, *i.e.* that

$$\triangle ABC \times \text{perpendicular from } B \text{ on } AC$$
$$= \frac{1}{3} \triangle ACD \times \text{perpendicular from } D \text{ on } AC,$$

i.e. that $$\frac{1}{2} AB \, . \, AC \sin 30^\circ \times AB \sin 30^\circ$$
$$= \frac{1}{3} \cdot \frac{1}{2} AD \, . \, AC \sin 60^\circ \times AD \sin 60^\circ,$$

which is true, since AB and AD are equal. Hence the quadrilateral will rest with AC vertical.

9. [Cf. Art. 75.] The forces having no resultant will not move the centre of gravity of the board, but will cause it to turn in its own plane round the peg, the couple causing rotation being measured by the area ABC.

10. The resultant of OB and OC must be in the direction OD, where D is the middle point of BC; hence, for equilibrium, AOD must be a straight line, or O lies on AD. Similarly, if E be the middle point of CA, O lies on BE; therefore O is at the intersection of AD and BE, which is the centre of gravity of the triangle ABC.

11. By Art. 42, the resultant of $\mu . PB$ and $\mu . PC$ is $2\mu . PD$, where D is the middle point of BC. Again, the resultant of $\mu . PA$ and $2\mu . PD$ is $3\mu . PG$, where $AG : GD = 2 : 1$, *i.e.* where G is the centre of gravity of the triangle ABC.

12. By Art. 42, $\lambda . PA$ and $\mu . PB$ have resultant $(\lambda + \mu) PQ$, where $AQ : QB = \mu : \lambda$. Join QC; the resultant of $(\lambda+\mu) PQ$ and $\nu . PC$ is $(\lambda+\mu+\nu) PR$, where $QR : RC = \nu : \lambda+\mu$; and so on. The construction for Q, R, &c. is the same as for the centre of gravity of weights at A, B, &c. proportional to λ, μ, &c.; hence the resultant is

$$(\lambda+\mu+\nu+\ldots) PG,$$

where G is the centre of gravity of such weights.

13. Let AB be the rod, G be its middle point, and O the point to which the strings are attached. Let the tensions of the strings be $\lambda . OA$ and $\mu . OB$. By Art. 42, they are equivalent to a force through a point in AB which divides it in the ratio $\lambda : \mu$. But, since their resultant balances the weight, it must go through the middle point of AB. Hence λ equals μ and so the tensions are proportional to the lengths.

In the second case let ABC be the triangle and let the tensions be $\lambda . OA$, $\mu . OB$, and $\nu . OC$ respectively. These, by the previous example, are equivalent to a force passing through a point in the triangle ABC which is the centre of gravity of weights proportional to λ, μ, and ν placed at A, B, and C.

But, since this resultant balances the weight, it must go through G, which is also the centre of gravity of equal weights placed at the angular points.

Hence λ, μ, and ν are equal, and therefore the tensions are proportional to the lengths.

14. Let A be the vertex of the cone, h be its height, l be its slant height, r be the radius of its base, and 2θ be the vertical angle. Then the area of the base $= \pi r^2$; the curved surface $= \pi rl$; and the whole surface $= \pi r(l+r)$. Hence, if G be the centre of gravity of the whole surface, we have

$$AG = \frac{\pi rl . \frac{2h}{3} + \pi r^2 . h}{\pi r(l+r)} = \frac{h}{3} . \frac{2l+3r}{l+r}$$

$$= \frac{h}{3} . \frac{2\sec\theta + 3\tan\theta}{\sec\theta + \tan\theta} = \frac{h}{3} . \frac{2+3\sin\theta}{1+\sin\theta} = \frac{3h}{4}, \text{ by hypothesis;}$$

$$\therefore\ 8 + 12\sin\theta = 9 + 9\sin\theta,$$

whence
$$\sin\theta = \frac{1}{3}.$$

$$\therefore\ 2\theta = 2\sin^{-1}\frac{1}{3}.$$

15. Let h be the height of the cone, h' be the height of the cylinder, A be the area of the common base, and G_1 and G_2 be the centres of gravity of the cylinder and the cone respectively. The volumes are $\frac{1}{3}hA$ and $h'A$, *i.e.* are as $h : 3h'$. Since the compound body balances about G, the centre of the common base, we have

$$G_1G : G_2G = \text{wt. at } G_2 : \text{wt. at } G_1,$$

i.e.
$$\frac{h'}{2} : \frac{h}{4} = h : 3h'.$$

$$\therefore \frac{h^2}{4} = \frac{3h'^2}{2}, \text{ so that } \frac{h^2}{h'^2} = 6,$$

and
$$h : h' = \sqrt{6} : 1.$$

16. Let O be the centre of the base, G be the vertex of the cone, h be the height of the cylinder, r be the radius of the base, and $OG = x$. The volumes of the cone and the cylinder are $\frac{1}{3}\pi r^2 x$ and $\pi r^2 h$. Hence, if G be the centre of gravity in the question, moments about O give

$$\pi r^2 h \cdot \frac{h}{2} = \frac{1}{3}\pi r^2 x \cdot \frac{x}{4} + \left(\pi r^2 h - \frac{1}{3}\pi r^2 x\right) x,$$

i.e.
$$\frac{h^2}{2} = \frac{x^2}{12} + hx - \frac{x^2}{3}.$$

$$\therefore x^2 - 4hx + 2h^2 = 0, \text{ whence } x = (2 \pm \sqrt{2})\, h;$$

the negative sign of the radical must be taken, as the positive sign would make $x > 2h$; hence

$$x = (2 - \sqrt{2})\, h,$$

i.e.
$$x : h = 2 - \sqrt{2} : 1.$$

17. Let ABC be the central section of the cone; then, if DEF be the inscribed circle, centre I, of ABC (touching the base BC in D, and AB and AC in E and F respectively), DEF will, by revolution, generate the greatest possible sphere, obviously. Now, since the centre of gravity of the remainder is to coincide with that of the cone, it is clear that the centre of gravity of the sphere must be at the same point. Let $BD = DC = a$, so that the base of the cone $BC = 2a$; then the height $AD = 2\sqrt{2}a$, and $AB = a\sqrt{1+8} = 3a$. Now, by Trigonometry,

$$ID = \frac{S}{s} = \frac{\frac{1}{2} \cdot 2 \cdot 2\sqrt{2}a^2}{\frac{1}{2}(3a + 3a + 2a)} = \frac{\sqrt{2}a}{2} = \frac{1}{4}AD,$$

i.e. the centre of the sphere does coincide with the centre of gravity of the cone; hence the centre of gravity of the remainder coincides with that of the cone.

Otherwise thus: If r be the radius of the sphere, we have

$$\sin BAD = \frac{FI}{AI} = \frac{r}{2\sqrt{2}a - r};$$

also $$\sin BAD = \frac{BD}{AB} = \frac{a}{3a} = \frac{1}{3};$$

$$\therefore \frac{r}{2\sqrt{2}a - r} = \frac{1}{3}, \text{ whence } r = \frac{\sqrt{2}a}{2} = \frac{1}{4}AD.$$

18. If h be the height of the cone, the radius of the base $= \frac{h}{\sqrt{3}}$, and the radius of the sphere $= \frac{h}{3}$. Thus the volume of the cone

$$= \frac{1}{3} \cdot \pi \cdot \frac{h^2}{3} \cdot h = \frac{\pi h^3}{9},$$

and the volume of the sphere

$$= \frac{4}{3} \cdot \pi \cdot \frac{h^3}{27} = \frac{4\pi h^3}{81}.$$

Let G be the centre of gravity of the cone, G_1 be that of the sphere, and G_2 be that of the remainder. Then, if A be the vertex of the cone, and D be the centre of the base, we have

$$DG = \frac{\text{wt. of sphere} \times DG_1 + \text{wt. of remainder} \times DG_2}{\text{wt. of cone}};$$

$$\therefore \frac{\pi h^3}{9} \cdot \frac{h}{4} = \frac{4\pi h^3}{81} \cdot \frac{h}{3} + \left(\frac{\pi h^3}{9} - \frac{4\pi h^3}{81}\right) DG_2.$$

$$\therefore \frac{9h}{4} = \frac{4h}{3} + 5DG_2,$$

whence $$DG_2 = \frac{11h}{60};$$

$$\therefore AG_2 = \frac{49h}{60},$$

i.e. G_2 divides the axis in the ratio $11 : 49$.

19. Let A be the vertex of the original cone, D be the centre of the base, C be the vertex of the hollow, G be the centre of gravity of the original cone, H be that of the hollow, and K be that of the remainder. The volume of the portion scooped out is one-half of the volume of the original cone, and therefore is equal to the volume of the remainder. We have

$$GK = GH = \frac{1}{4} DC;$$

$$\therefore DK = \frac{3}{4} DC = \frac{3}{8} AD, \text{ and } KA = \frac{5}{8} AD;$$

$$\therefore DK : KA = 3 : 5.$$

20. Let the $\angle BAD=\theta=$ the $\angle BDA$; O be the point in which the lines of action of R the horizontal reaction at A, W the weight of the triangle, and T the tension of the string meet; let G be the centre of gravity of the triangle.

Then we have $$OA=AD\tan\theta,$$

and $$OA=AG\cos OAG=AG\sin GAD;$$

$$\therefore\ AD\tan\theta=AG\sin GAD \qquad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the $\angle GAD=$ the $\angle GAB+\theta=30^\circ+\theta$, and, if a be the side of the triangle,

$$AD=2a\cos\theta,\ \text{and}\ AG=\frac{a}{2}\sec GAB,$$

hence, from (1), $$2a\sin\theta=\frac{a}{2\cos 30^\circ}\cdot\sin(30^\circ+\theta),$$

i.e. $$2\sqrt{3}\sin\theta=\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta;$$

whence $$\sin\theta=\frac{1}{\sqrt{28}},\ \text{and}\ \cos\theta=\frac{3\sqrt{3}}{\sqrt{28}}.$$

Again, if h and k be the distances of B and C from AD, we have

$$h=a\sin\theta,\ \text{and}\ k=a\sin(60^\circ+\theta);$$

hence $$\begin{aligned} h:k&=\sin\theta:\sin(60^\circ+\theta)\\ &=\frac{1}{\sqrt{28}}:\frac{\sqrt{3}}{2}\cdot\frac{3\sqrt{3}}{\sqrt{28}}+\frac{1}{2}\cdot\frac{1}{\sqrt{28}}\\ &=2:10=1:5.\end{aligned}$$

Otherwise thus: Resolving vertically, we have $W=T\cos\theta$; and replacing the weight W by three equal weights $\frac{W}{3}$ at the angular points A, B and C, moments about A give

$$\frac{W}{3}\cdot h+\frac{W}{3}\cdot k=T\,.\,AB\sin 2\theta=a\,.\,2W\sin\theta;$$

hence $$h+k=6a\sin\theta;\ \text{but}\ h=a\sin\theta,$$

so that $$k=5a\sin\theta,\ \text{and}\ h:k=1:5.$$

21. If P be the point in the circumference of the base, A be the vertex, AH be the axis, and G be the centre of gravity of the cone, then PG is vertical; also, since $AH=4PH$, and $AH=4HG$, therefore $HG=HP$. Hence the angles HGP and HPG which the axis and the base make with the vertical are equal.

22. Let A be the common vertex, AB be the common slant side of length l, and G_1 and G_2 be the centres of gravity of the two cones respectively. Then

the $\angle G_1AB=30^\circ$, the $\angle G_2AB=60^\circ$, and the $\angle G_1AG_2=90^\circ$.

The radii of the bases are

$$l \sin 30^\circ \text{ and } l \sin 60^\circ,$$

i.e.
$$\frac{l}{2} \text{ and } \frac{l\sqrt{3}}{2};$$

the heights are $l \cos 30^\circ$ and $l \cos 60^\circ$, *i.e.* $\frac{l\sqrt{3}}{2}$ and $\frac{l}{2}$.

Therefore, if V_1 and V_2 be the volumes, we have

$$V_1 = \frac{1}{3} \cdot \pi \cdot \frac{l^2}{4} \cdot \frac{l\sqrt{3}}{2} \text{ and } V_2 = \frac{1}{3} \cdot \pi \cdot \frac{3l^2}{4} \cdot \frac{l}{2};$$

$$\therefore\ V_1 : V_2 = \frac{\sqrt{3}}{8} : \frac{3}{8} = 1 : \sqrt{3}.$$

Hence, if G be the centre of gravity of both cones,

$$G_1G : G_2G = \sqrt{3} : 1;$$

also
$$AG_1 : AG_2 = \cos 30^\circ : \cos 60^\circ = \sqrt{3} : 1;$$

$$\therefore\ G_1G : G_2G = AG_1 : AG_2.$$

$$\therefore \text{ the } \angle G_1AG = \frac{1}{2} \angle G_1AG_2 = 45^\circ.$$

$\therefore$ AB makes with the vertical AG the angle $(45^\circ - 30^\circ)$, *i.e.* 15°.

23. Let ABC be the piece of paper, A being the vertex, and let F and E be the middle points of AB and AC respectively. Draw AD perpendicular to BC; then when folded across the line DE, A will rest on BC at D. Also, if AD be h, the distance from BC of the centre of inertia of ABC when unfolded is $\frac{h}{3}$; and if the distance be x when folded over, we have

$$x = \frac{\triangle BDF \cdot \frac{h}{6} + \triangle DEC \cdot \frac{h}{6} + 2\triangle DEF \cdot \frac{h}{3}}{\triangle ABC}$$

$$= \frac{h}{6} \cdot \frac{\triangle BDF + \triangle DEC + 4\triangle DEF}{\triangle ABC}$$

$$= \frac{h}{6} \cdot \frac{\triangle BDF + \triangle DEC + 2\triangle DEF + 2\triangle DEF}{\triangle ABC}$$

$$= \frac{h}{6} \cdot \frac{\triangle ABC + \frac{1}{2}\triangle ABC}{\triangle ABC}$$

$$= \frac{h}{6} \cdot \frac{3}{2} = \frac{h}{4} = \frac{3}{4} \cdot \frac{h}{3}.$$

24. Let $ABCD$ be the sheet of paper originally, with the sides AB and CD each $=2a\sqrt{2}$, and the sides AD and BC each $=2a$. Let E and F be the middle points of AD and BC respectively, and let the corners A and D be doubled over to L, and HK be the edge of the table. Then G, the new centre of gravity, is to be at the centre of HK. Let MN be the line through L parallel to AD. Then we have

$$\triangle ELM=\frac{a^2}{2},$$

and hence $$4\triangle ELM=2a^2.$$

Also $$LF=2a\sqrt{2}-a=a\,(2\sqrt{2}-1),$$

and area $MBCN=$ area $ABCD-4\triangle ELM=4a^2\sqrt{2}-2a^2=2a^2(2\sqrt{2}-1)$

Hence

$$EG=\frac{4\triangle ELM\times\frac{2}{3}EL+\text{area }MBCN\left(EL+\frac{1}{2}LF\right)}{\text{area }ABCD}$$

$$=\frac{2a^2\times\frac{2}{3}a+2a^2\,(2\sqrt{2}-1)\left(a+a\sqrt{2}-\frac{a}{2}\right)}{4a^2\sqrt{2}}$$

$$=\frac{a}{4\sqrt{2}}\left[\frac{4}{3}+(2\sqrt{2}-1)\,(2\sqrt{2}+1)\right]=\frac{a}{4\sqrt{2}}\cdot\frac{25}{3}$$

$$=\frac{25}{48}\,(2a\sqrt{2})=\frac{25}{48}AB;$$

this is the length on the table.

25. If O be the centre of the circumscribing circle, A be the angle at which no particle is placed, and G be the centre of gravity of the $(n-1)$ particles, then if an equal particle were placed at A, the centre of gravity of the whole would be at O. Hence AOG is a straight line, and we have

$$OG:OA=1:n-1,\quad i.e.\ OG=\frac{r}{n-1}.$$

26. The area of the circle$=\dfrac{\pi a^2}{4}$, and the area of the square

$$=\frac{1}{2}\cdot\frac{a^2}{4}=\frac{a^2}{8}.$$

Let G be the centre of the circle, G_1 be the centre of the square, and G_2 be the centre of gravity of the remainder.

Then $$GG_2:GG_1=\text{wt. at }G_1:\text{wt. at }G_2$$

$$=\text{wt. of square}:\text{wt. of remainder}=\frac{a^2}{8}:\frac{\pi a^2}{4}-\frac{a^2}{8}$$

$$=1:2\pi-1;$$

$$\therefore\ GG_2=\frac{GG_1}{2\pi-1}=\frac{a}{4}\div(2\pi-1)=\frac{a}{8\pi-4}.$$

27. Let ABC be the board, G be its centre of gravity, I be the centre of the inscribed circle, and H be the centre of gravity of the remainder after the portion is removed. Draw IN (which is r, the radius of the inscribed circle), GL, and HM perpendicular to BC. Then

$$GL=\frac{p}{3},$$

where p is the perpendicular distance of A from BC,

$$=\frac{1}{3}\cdot\frac{2S}{a}.$$

Also, we have

$$GL\,.\,S=IN\,.\,\pi r^2+HM\,(S-\pi r^2).$$

$$\therefore\ \frac{2S^2}{3a}=\pi r^3+HM\,(S-\pi r^2);$$

hence, since $$r=\frac{S}{s},$$

$$HM=\left[\frac{2S^2}{3a}-\pi\left(\frac{S}{s}\right)^3\right]\div\left[S-\pi\left(\frac{S}{s}\right)^2\right],$$

$$=\frac{S}{3as}\cdot\frac{2s^3-3\pi aS}{s^2-\pi S}.$$

28. Let R and r be the radii of the plate and the hole respectively. The centre of gravity of the remainder is farthest from the centre of the original circle when the piece punched out is close to the circumference of the plate. Let G, G_1, and G_2 be the centres of gravity of the whole plate, the hole, and the remainder respectively. Then

$$GG_2 : GG_1=\text{wt. at } G_1 : \text{wt. at } G_2=\pi r^2 : \pi R^2-\pi r^2;$$

$$\therefore\ GG_2=GG_1\,.\,\frac{r^2}{R^2-r^2}=(R-r)\,\frac{r^2}{R^2-r^2}=\frac{r^2}{R+r},$$

so that G_2 can lie anywhere on or within the circle whose centre is G and radius $\frac{r^2}{R+r}$.

30. Join A and B. Then

$$AG : GB=n : m,$$

and therefore $$m\,.\,AG=n\,.\,BG.$$

We have

$$m\,.\,AP^2+n\,.\,BP^2=m\,(AG^2+PG^2-2AG\,.\,PG\cos AGP)$$
$$+n\,(BG^2+PG^2-2BG\,.\,PG\cos BGP);$$

now $$\cos AGP=-\cos BGP;$$

hence $$m\,.\,AP^2+n\,.\,BP^2=m\,.\,AG^2+n\,.\,BG^2+(m+n)\,PG^2$$

For the general case draw through G, as in Art. 111, two lines at right angles, GX and GY, and let the distances of the points A, B, C,... measured along these lines be x_1, x_2, x_3... and y_1, y_2, y_3.... Also let the distances of P be x and y respectively.

By Art. 111 we have

$$\frac{mx_1+nx_2+\dots}{m+n+\dots}=\text{distance of } G \text{ measured along } GX=0 \dots\dots(1).$$

So
$$\frac{my_1+ny_2+\dots}{m+n+\dots}=0 \dots\dots\dots\dots\dots\dots (2).$$

Hence

$$m\,.\,AP^2+n\,.\,BP^2+\dots=m\{(x-x_1)^2+(y-y_1)^2\}$$
$$+n\{(x-x_2)^2+(y-y_2)^2\}+\dots$$
$$=m\{(x^2+y^2)+(x_1^2+y_1^2)\}+n\{(x^2+y^2)+(x_2^2+y_2^2)\}+\dots$$
$$-2x(mx_1+nx_2+\dots)-2y(my_1+ny_2+\dots)$$
$$=m\{GP^2+GA^2\}+n\{GP^2+GB^2\}+\dots$$

by using equations (1) and (2),

$$=m\,.\,AG^2+n\,.\,BG^2+\dots+(m+n+p+\dots)PG^2.$$

EXAMPLES. XX. (Pages 159—162.)

1. Let ABC be the rule, B be the angular point, and AB be the longer part. The weights of the parts are proportional to their lengths, *i.e.* to 2 and 1, and act at E and D the middle points of AB and BC respectively. Join ED; then the centre of gravity is at G, where $EG=\frac{1}{3}ED$. Also, if GF be the vertical line from G, meeting BC in F, we have

$$BF=\frac{1}{3}BD=\frac{1}{3}\times 4=1\tfrac{1}{3} \text{ ins.},$$

and
$$FC=8-\frac{4}{3}=6\tfrac{2}{3} \text{ ins.};$$

the portion FC at least must rest on the table.

2. If h be the height of the cylinder, and r be the radius of its base, then

$$\tan 30°=\frac{2r}{h},\ \textit{i.e.}\ r=\frac{h}{2\sqrt{3}};$$

but
$$\pi r^2 h=18 \text{ cub. ins.},$$

so that
$$\pi\times\frac{h^2}{12}\times h=18 \text{ cub. ins.}$$

$$\therefore h^3=\frac{216}{\pi}, \text{ and } h=\frac{6}{\sqrt[3]{\pi}} \text{ ins.}$$

3. Let ABC be the lamina and D the middle point of BA the edge on which it stands. Let the vertical line from C meet BA produced in E; join CD. Then, if G be the centre of gravity of the lamina,

$$DG=\frac{1}{2}GC;$$

hence, since G must be just vertically over A, we have

$$DA=\frac{1}{2}AE,$$

so that $$AE=BA.$$

Now $$BC^2=BE^2+EC^2,$$

i.e. $$a^2=4c^2+CA^2-AE^2$$
$$=4c^2+b^2-c^2=b^2+3c^2.$$

4. Let the square $ABCD$ be divided into two equal parts by the line EF; then the part $AFED$ is divided by AE into the two equal triangles AEF and AED. The weight of the part $EFBC$ is $\frac{W}{2}$ acting at G, its middle point, and of the triangle AEF is $\frac{W}{4}$ acting at H, where

$$EH=\frac{2}{3}EK,$$

K being the middle point of AF. Hence, if X be the required weight, moments about E give

$$X.\frac{AB}{2}+\frac{W}{4}.\frac{2}{3}.\frac{AB}{4}=\frac{W}{2}.\frac{AB}{4},$$

i.e. $$\frac{X}{2}+\frac{W}{24}=\frac{W}{8}, \text{ whence } X=\frac{W}{6}.$$

5. Since D is the middle point of AC, if DE be drawn parallel to AB (which is vertical) to meet BC in E, then E will be the middle point of BC; hence the centre of gravity of the triangle BDC is in DE, and, DE being vertical, the triangle BDC is just on the point of falling over.

6. Let O be the middle point of the side on the wall of the lowest brick of four courses; let OX be a fixed line perpendicular to the edge of the wall. If l be the length and W the weight of a brick we have

$$\bar{x}=\frac{W.0+\frac{5W}{4}.\frac{l}{8}+\frac{3W}{2}.\frac{l}{4}+\frac{7W}{4}.\frac{3l}{8}}{W+\frac{5W}{4}+\frac{3W}{2}+\frac{7W}{4}}=\frac{19l}{88},$$

i.e. $$\bar{x}<\frac{l}{4},$$

so that the vertical line through the centre of gravity of the four courses meets the wall; hence there is equilibrium.

If a fifth course be added, we must add a term $2W.\frac{l}{2}$ to the numerator, and $2W$ to the denominator in the above value of $\bar{x}$; we then have

$$\bar{x}=\frac{7l}{24}, \text{ i.e. } \bar{x}>\frac{l}{4},$$

and the structure will now topple.

7. Let h be the height of the pile, d be the diameter of each coin, and a be the inclination of the plane. Since the pile is just on the point of toppling the centre of gravity of the pile must be just vertically over the lowest point of the base. Hence

$$\frac{d}{h}=\tan a=\frac{1}{6};$$

$$\therefore h=6d=120\times\frac{d}{20},$$

and the number of coins is 120. If the pile be of unlimited height, the centres of gravity of the coins must form a vertical line. Let A and B be two centres of gravity, and BC the thickness $\left(=\frac{d}{20}\right)$. Then

$$AC=\text{amount of overlap}=BC\tan a=\frac{d}{20}\times\frac{1}{6}$$

$$=\frac{1}{120}\text{th of diameter.}$$

8. Let AB ($=9$ ins.) be the length of the side on the table and let ABX be the fixed line. Let the number of bricks above the lowest brick be n. The distance from A of the centre of gravity of these n bricks is given by

$$\bar{x}=\frac{5+5\frac{1}{2}+6+\dots\text{ to } n \text{ terms}}{1+1+1+\dots\text{ to } n \text{ terms}}=\frac{\frac{n}{2}\left[10+\frac{1}{2}(n-1)\right]}{n}=\frac{n+19}{4}.$$

The structure will stand provided that $\bar{x}$ be not greater than 9, *i.e.* if $n+19$ be not greater than 36. The greatest value of n is therefore 17 and the total possible number of bricks is $17+1$, *i.e.* 18.

If the bricks be placed the other way, so that $AB=4$, we have

$$\frac{2\frac{1}{2}+3\frac{1}{2}+\dots\text{ to } n \text{ terms}}{1+1+\dots\text{ to } n \text{ terms}}$$

not greater than 4, so that $\frac{n+9}{4}$ is not greater than 4. The greatest value of n is therefore 7 and the required number is 8.

9. By Art. 104 the triangle can be replaced by 3 weights, each equal to $\frac{W}{3}$, placed at the angular points. When the additional weight is put on, we have to shew that a weight $\frac{2W}{3}$ at C and a weight $\frac{W}{3}$ at B just balance about A.

If CN be drawn perpendicular to BA, then

$$AN = AC\cos 60^\circ = AB\cos 60^\circ = \frac{1}{2}\,.\,AB.$$

Hence $$\frac{2W}{3}\times AN = \frac{W}{3}\times AB,$$
or the weights just balance about A.

10. If E be the middle point of AC, we have

$$\triangle ABC : \triangle ADC = BE : ED.$$

Also BED is a straight line perpendicular to AC. Let AB and BC be taken as the two fixed lines from which the distances are measured. Then we have

$$\bar{x} = \frac{\triangle ABC \times \frac{2}{3}BE\cos 45^\circ + \triangle ADC\left(BE + \frac{1}{3}ED\right)\cos 45^\circ}{\triangle ABC + \triangle ADC}$$

$$= \frac{\frac{2}{3}BE^2 + ED\left(BE + \frac{1}{3}ED\right)}{BE + ED}\cdot\frac{1}{\sqrt{2}},$$

which is to be $\quad < BC$, *i.e.* $< BE\sqrt{2}$;

$$\therefore\ \frac{2}{3}BE^2 + BE\,.\,ED + \frac{1}{3}ED^2 < 2BE^2 + 2BE\,.\,ED,$$

i.e. $$\frac{1}{3}ED^2 - BE\,.\,ED < \frac{4}{3}BE^2,$$

i.e. $$ED^2 - 3BE\,.\,ED - 4BE^2 < 0,$$

i.e. $$(ED - 4BE)(ED + BE) < 0,$$

i.e. $$ED < 4BE,$$

i.e. $$\triangle ADC < 4\triangle ABC.$$

11. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively; h be the height of the cone, and r be the radius of the base. For neutral equilibrium, the centre of gravity of the compound body must be at O the centre of the base of the hemisphere. Also the weights of the hemisphere and cone balance about O, and are proportional to their volumes, *i.e.* to

$$\frac{2}{3}\pi r^3 \text{ and } \frac{1}{3}\pi r^2 h;$$

also $$OG_1=\frac{3}{8}r, \text{ and } OG_2=\frac{h}{4}.$$

Hence we have $$\frac{1}{3}\pi r^2 h\cdot\frac{h}{4}=\frac{2}{3}\pi r^3\cdot\frac{3}{8}r,$$

whence $$h^2=3r^2, \text{ and } h=r\sqrt{3}.$$

For the equilibrium to be stable,

$$h<r\sqrt{3}.$$

12. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cylinder respectively; h be the height of the cylinder, and r be the radius of the base. For neutral equilibrium, the centre of gravity of the compound body must be at O the centre of the base of the hemisphere. Also the weights of the hemisphere and cylinder balance about O, and are proportional to their volumes, *i.e.* to

$$\frac{2}{3}\pi r^3 \text{ and } \pi r^2 h;$$

also $$OG_1=\frac{3}{8}r, \text{ and } OG_2=\frac{h}{2}.$$

Hence we have $$\pi r^2 h\cdot\frac{h}{2}=\frac{2}{3}\pi r^3\cdot\frac{3}{8}r,$$

whence $$h^2=\frac{r^2}{2}, \textit{ i.e. } h:r=1:\sqrt{2}.$$

13. With the notation of Art. 126, we have

$$(1)\quad h=\frac{5}{8}r, \text{ and } R=r;$$

$$\therefore \frac{1}{h}=\frac{8}{5}\cdot\frac{1}{r}, \text{ and } \frac{1}{r}+\frac{1}{R}=\frac{2}{r};$$

hence $\frac{1}{h}<\frac{1}{r}+\frac{1}{R}$, and the equilibrium is unstable.

$$(2)\quad h=\frac{3}{8}R, \text{ and } r=\infty;$$

$$\therefore \frac{1}{h}=\frac{8}{3}\cdot\frac{1}{R}, \text{ and } \frac{1}{r}+\frac{1}{R}=\frac{1}{R}+0;$$

hence $\frac{1}{h}>\frac{1}{r}+\frac{1}{R}$, and the equilibrium is stable.

14. If AB $(=h)$ be the height of the cone, and G be its centre of gravity, then $AG=\frac{h}{4}$.

By Cor. 2, p. 159,

$$\frac{h}{4}<R, \textit{ i.e. } h<4R.$$

15. By Cor. 2, p. 159, the equilibrium is stable or unstable, according as b is $<$ or $>$ a.

16. Let A be the highest point of the sphere, O be its centre, and G be the centre of gravity of the cube. Then, when the cube balances on A, GA is vertical and $=\frac{1}{2}\cdot\frac{\pi r}{2}=\frac{\pi r}{4}$, which is $<r$; hence the equilibrium is stable.

Let the cube roll over an arc AC, where

$$\angle AOC=\frac{\pi}{4}, \text{ i.e. } \operatorname{arc} AC=r\theta=\frac{\pi r}{4};$$

in this position the vertical through G passes through C; and since half of the side of the cube $=\frac{1}{2}\cdot\frac{\pi r}{2}=\frac{\pi r}{4}$, we see that the cube can rock through 45° on either side of A, *i.e.* can rock through a right angle, without falling.

17. Let ABC be the lamina, and D be the point in its side BA in contact with the sphere. If G be the centre of gravity of the lamina, GD is vertical. We have

$$AG=\frac{2}{3}\cdot a\cos\frac{\alpha}{2};$$

also $$GD=AD\sin\frac{\alpha}{2},$$

i.e. $$GD=\frac{2}{3}\cdot a\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\frac{a}{3}\sin\alpha=h;$$

the equilibrium is stable if $h<r$ [Cor. 2, p. 159],

i.e. if $$\frac{a}{3}\sin\alpha<r,$$

i.e. if $$\sin\alpha<\frac{3r}{a}.$$

18. If α be the inclination of the plane to the horizon, and θ be the angle between the string and the plane, then resolving along the plane (the tension of the string being equal to P), we have

$$W\sin\alpha=P\cos\theta,$$

i.e. $$\theta=\cos^{-1}\left(\frac{W\sin\alpha}{P}\right).$$

Knowing θ, and the position of the pulley, that of W can be found.

Again, if W were moved slightly down the plane, θ would be diminished, and, therefore, $P\cos\theta$ increased; thus W would move up again. If W be moved slightly up the plane, θ is increased, and, therefore, $P\cos\theta$ diminished; thus W would move down again. Hence the equilibrium is stable.

19. If O be the centre of the disc, C be the fixed point, and CO make an angle θ with the vertical, moments about C give

$$W(r - c\sin\theta) = w(r + c\sin\theta) + pc\sin\theta,$$

whence $$\sin\theta = \frac{W - w}{p + W + w} \cdot \frac{r}{c},$$

i.e. $$\theta = \sin^{-1}\left[\frac{W - w}{p + W + w} \cdot \frac{r}{c}\right].$$

The moment about C to increase θ being

$$W(r - c\sin\theta) - w(r + c\sin\theta) - pc\sin\theta,$$

which decreases or increases according as θ increases or decreases, the equilibrium is stable.

20. Let w be the weight of the solid sphere whose radius is r. Let W be the weight placed at the highest point so that the height h of the combined C.G. above the lowest point

$$= \frac{w \cdot r + W \cdot 2r}{w + W}.$$

By Art. 126, Cor. 1, the equilibrium is stable if

$$\frac{1}{h} > \frac{1}{r} - \frac{1}{2r}, \text{ i.e. if } 2r > h,$$

i.e. if $$2r > \frac{w + 2W}{w + W} r,$$

i.e. if $$2w + 2W > w + 2W,$$

which is always the case.

21. When the bowl is displaced and held still, the small sphere moves also in such a way that its weight still passes through the centre of the bowl. As far as the problem of stability is concerned the weight of the small sphere may thus be taken to act at the centre of the bowl.

The height h of the combined C.G. therefore

$$= \frac{w \cdot b + W \cdot \frac{b}{2}}{w + W}.$$

There is stability (Art. 126) if

$$\frac{1}{h} > \frac{1}{a} + \frac{1}{b},$$

i.e. if $$\frac{w + W}{w + \frac{1}{2}W} \frac{1}{b} > \frac{1}{a} + \frac{1}{b},$$

i.e. if $$(w + W)a > (a + b)(w + \tfrac{1}{2}W),$$

i.e. if $$\tfrac{1}{2}W \cdot a > b(w + \tfrac{1}{2}W),$$

i.e. if $$w < \frac{a - b}{2b} W.$$

EXAMPLES. XXI. (Pages 168—170.)

1. (1) *Ans.* $=$ wt. $\times$ ht. climbed

$$= 140 \times 2700 \text{ ft.-lbs.} = \frac{140 \times 2700}{2240} \text{ ft.-tons}$$

$$= 168{\cdot}75 \text{ ft.-tons.}$$

(2) *Ans.* $=$ Distance $\times$ Resistance

$$= 52800 \times 5 \text{ ft.-lbs.} = \frac{264000}{2240} \text{ ft.-tons}$$

$$= 117\tfrac{6}{7} \text{ ft.-tons.}$$

2. If x feet be the length of the chain hanging down the shaft, the centre of gravity is raised a height $\frac{x}{2}$ feet; hence

$$8x \times \frac{x}{2} = 4000000 \text{ ft.-lbs.},$$

i.e. $$4x^2 = 4000000 \text{ ft.-lbs.},$$

and $$x = 1000 \text{ ft.}$$

3. Weight of the soil $= 10 \times 8 \times 100 \times 150$ lbs.
Distance through which the C.G. is raised $= 50$ ft.
$\therefore$ Work done $=$ product $= 10 \times 8 \times 100 \times 150 \times 50$ ft.-lbs.

$$= 6 \times 10^7 \text{ ft.-lbs.}$$

4. A cubic foot of water weighing $\frac{1000}{16}$ lbs., the required number of cubic feet

$$= \frac{100 \times 33000 \times 60}{\left(\frac{1000}{16} \times 150\right)} = 21120.$$

5. The number of hours being x, and the depth of the centre of gravity of the water being $\frac{420}{2}$ feet, *i.e.* 210 feet, we have

$$18 \times 33000 \times 60 \times x = \pi \times \left(\frac{9}{2}\right)^2 \times 420 \times \frac{1000}{16} \times 210,$$

whence $$x = 9\tfrac{27}{32} \text{ hours.}$$

6. Here, the required H.P.

$$= \frac{\pi \times 4^2 \times 600 \times 300}{33000 \times 32 \times 60} \times \frac{1000}{16} = 8\tfrac{13}{14}.$$

7. If the time be t minutes, then

$$t \times \frac{2}{3} \times 20 \times 33000 = 5000 \times 62\tfrac{1}{2} \times 100,$$

since a cubic foot of water weighs $62\frac{1}{2}$ lbs.

$$\therefore\ t = \frac{25 \times 125}{44} = 71\tfrac{1}{44}.$$

8. If x be the required H.P., then

$$x \times 550 = \text{work done per sec.}$$
$$= 140 \times 1\tfrac{1}{2}.$$
$$\therefore\ x = \frac{210}{550} = \frac{21}{55}, \text{ i.e. } x \text{ is just} < \frac{2}{5}.$$

9. The volume of the tower

$$= (22 \times 9 \times 66 - 18 \times 5 \times 66) \text{ cub. ft.}$$
$$= 108 \times 66 \text{ cub. ft.};$$

the weight of the brickwork $= (108 \times 66 \times 112)$ lbs.;
the height of the centre of gravity $= 33$ feet;
the work done $= (108 \times 66 \times 112 \times 33)$ ft.-lbs.;
but the engine does (3×33000) ft.-lbs. per min.; hence the required number of hours

$$= \frac{108 \times 66 \times 112 \times 33}{3 \times 33000 \times 60} = 4{\cdot}4352.$$

10. The wire rope weighs 2 lbs. per foot, and its centre of gravity is raised through $\frac{275}{2}$ feet, its length being 275 feet. Hence the work done

$$= \left[(18 \times 112 + 109)\, 275 + 2 \times 275 \times \frac{275}{2}\right] \text{ ft.-lbs.}$$
$$= 660000 \text{ ft.-lbs.}$$

Also, the required H.P.

$$= \frac{660000}{33000 \times \frac{2}{3}} = 30.$$

11. 15 miles per hour $= 22$ ft. per sec.
Hence, if P be the resistance in lbs. wt.,

$$P \times 22 = 10000 \times 550.$$
$$\therefore\ P = 250000 = \frac{250000}{2240} \text{ tons' wt.}$$
$$= 111\tfrac{17}{28} \text{ tons' wt.}$$

12. 6 miles per hour $=\frac{88}{10}$ ft. per sec.

Resolved part of wt. down the hill $=\frac{1}{20}\times 200$ lbs.

$=10$ lbs.

$\therefore$ work done per sec. $=\frac{88}{10}\times 10=88$ ft.-lbs.

$\therefore$ H.P. to do this $=\frac{88}{550}=\frac{8}{50}=\cdot 16$.

13. If W be the required work, then

$40W=8\times 880$ (since 10 miles per hour $=880$ ft. per min.).

$\therefore\ W=\frac{7040}{40}=176$.

Also the H.P. $=\frac{\text{work done per minute}}{33000}$

$=\frac{880\times 8}{33000}=\cdot 21\dot{3}$ H.P.

14. Total weight of the bars $=\frac{30\times 4}{16}=7\frac{1}{2}$ lbs.

Depth of centre of gravity originally

$$=\frac{(1+2+3+\ldots+30)\times 2\frac{1}{2}}{30}=\frac{\frac{1}{2}.30\;31.2\frac{1}{2}}{30}$$

$$=\frac{155}{4}\text{ inches.}$$

$\therefore$ work done $=7\frac{1}{2}\times\frac{155}{48}$ ft.-lbs.

$=\frac{2325}{96}$ ft.-lbs. $=24\cdot 2$ ft.-lbs.

In the second case total wt. $=\frac{n}{4}$ lbs.

Depth of C.G. $=\frac{(1+2+3+\ldots+n)\times 2\frac{1}{2}}{n}$

$$=\frac{\frac{1}{2}n.(n+1)\times 2\frac{1}{2}}{n}=\frac{5}{4}(n+1)\text{ inches}=\frac{5}{48}(n+1)\text{ ft.}$$

$\therefore$ work done

$$=\frac{n}{4}\times\frac{5}{48}(n+1)=\frac{5}{192}n(n+1)\text{ ft.-lbs.}$$

15. The original distance between each bar $=\frac{a}{n}$.

Hence depth of the c.g. originally

$$=\frac{(1+2+3+\ldots+n)\times\frac{a}{n}}{n}=\frac{\frac{1}{2}n(n+1)\times\frac{a}{n}}{n}=(n+1)\frac{a}{2n}.$$

Final depth $=(n+1)\frac{b}{2n}$.

$\therefore$ Work done $=W\times$ difference of these distances.

16. By Art. 117 the heights of its centre of gravity above the table in its initial and final positions are $\frac{3}{8}$ and $\frac{5}{8}$ ft. respectively.

$$\therefore \text{ Work done}=12 \text{ lbs.}\times\tfrac{2}{8}\text{ ft.}$$
$$=3 \text{ ft.-lbs.}$$

17. Let ABC be the section of the prism so that $BC=1\frac{1}{2}$, $CA=2$, and $AB=2\frac{1}{2}$ feet.

Let D be the middle point of BC, G the centre of gravity and GN the perpendicular on BC. Then, since ACB is a right angle, we have

$$GN=\frac{1}{3}AC=\frac{2}{3},\quad\text{and}\quad DN=\frac{1}{3}DC=\frac{1}{4}.$$

$$\therefore\ BN=\frac{3}{4}+\frac{1}{4}=1\quad\text{and}\quad BG=\sqrt{1+\frac{4}{9}}=\frac{\sqrt{13}}{3}.$$

The centre of gravity has to be raised through a height $BG-GN$, *i.e.* through $\frac{\sqrt{13}-2}{3}$ feet.

$$\therefore\ \text{work}=\frac{1}{2}\times\frac{\sqrt{13}-2}{3}\text{ ft.-tons}$$
$$=\frac{1{\cdot}6056}{6}=\cdot2676\text{ ft.-tons.}$$

18. Let $ABCDEFG$ be a straight line such that

$$AB=BC=CD=DE=EF=FG=\text{one foot.}$$

Erect perpendiculars AA_1, BB_1, ... GG_1 equal respectively to 20, 25, 29, ... 24. Then the area of the figure $AA_1B_1C_1D_1E_1F_1G_1GA$ represents the number of foot-lbs. required.

The area of the trapezium AA_1B_1B

$$=\tfrac{1}{2}AB(AA_1+BB_1)=\tfrac{1}{2}(20+25).$$

So for the other trapezia. Hence the total area

$$=\tfrac{1}{2}(20+25)+\tfrac{1}{2}(25+29)+\tfrac{1}{2}(29+32)+\tfrac{1}{2}(32+31)$$
$$+\tfrac{1}{2}(31+27)+\tfrac{1}{2}(27+24)=166\text{ ft.-lbs.}$$

EXAMPLES. XXII. (Pages 178—180.)

1. Take the first figure of Page 174, with $CB=3$ feet, and the force of 10 lbs. wt. acting at B. Then

$$P+10=R=16 \text{ lbs. wt.},$$

so that

$$P=6 \text{ lbs. wt.};$$

also moments about C give $6\times AC=10\times 3$, whence $AC=5$ feet.

2. With the same figure, let the weight of 6 lbs. be at A, and the weight of 8 lbs. at B. If AC be x feet, then BC is $(7-x)$ feet, and moments about C give $6\times x=8(7-x)$, whence $x=4$ feet.

Again, the moment of the weight of 7 lbs. is (7×4) ft.-lbs., *i.e.* 28 ft.-lbs.; and the moment of the weight of 9 lbs. is (9×3) ft.-lbs., *i.e.* 27 ft.-lbs.; hence the lever will turn towards the weight of 7 lbs.

3. With the same figure, let the two forces at A and B be P and Q respectively; let R be the pressure, and a and b be the lengths of the arms AC and CB respectively. Then we have

$$P+Q=R, \text{ and } Pa=Qb;$$

also

$$R=10(P-Q),$$

i.e.

$$P+Q=10(P-Q).$$

$$\therefore \frac{P+Q}{P-Q}=\frac{10}{1}; \text{ hence } \frac{P}{Q}=\frac{11}{9}=\frac{b}{a}.$$

4. The lever being supposed uniform and of weight W lbs., moments about the point, at distance of 9 ins. from the end where the weight of 20 lbs. is fastened, give

$$6\times 27+W\times 9=20\times 9,$$

whence

$$W=2 \text{ lbs.}$$

5. If W be the weight of the lever, acting at G the required centre of gravity, we have

$$W(GA-1)=13\times 1,$$

and
$$W(GB-1)=11\times 1;$$

$$\therefore\ 11(GA-1)=13(GB-1),$$

i.e.
$$11GA-13(12-GA)=-2,$$

whence
$$GA=\frac{77}{12}\text{ feet}=6\text{ feet }5\text{ ins.};$$

hence, since $\frac{AB}{2}=6$ feet, G is 5 ins. from the middle point of the lever, towards B.

6. Let l be the length of the lever, and W lbs be its weight acting at its middle point. Then moments about the fulcrum, which is at distance $\frac{l}{3}$ from the end at which the weight of 12 lbs. is attached, give

$$12\times\frac{l}{3}=W\times\frac{l}{6}+5\times\frac{2l}{3},$$

whence
$$W=4\text{ lbs.}$$

7. [Cf. the second figure, p 174.] The weight of the lever acts at its middle point. If P lbs. wt. be the (vertical) force at the other end, moments about the fulcrum give

$$3\times 1+6\times 3+10\times\frac{5}{2}=P\times 5,$$

whence
$$P=9\tfrac{1}{5}\text{ lbs. wt.};$$

and the pressure on the fulcrum

$$=(3+10+6)-P=9\tfrac{4}{5}\text{ lbs wt.}$$

8. The weight of the lever acts at its middle point. Moments about the fulcrum, which is at distance of x ins. from the weight of 27 oz., give

$$27x=18(9-x)+9(18-x),$$

whence
$$x=6\text{ ins.}$$

In the second case, if the fulcrum be at distance of y ins. from the weight of 27 oz., we have

$$27y=18(9-y)+18(18-y),$$

whence
$$y=7\tfrac{5}{7}\text{ ins.};$$

hence the position of the fulcrum must be shifted $1\tfrac{5}{7}$ in. towards the middle point of the lever.

9. The fulcrum divides the rod in the ratio of 4 : 8, *i.e.* 1 : 2. On adding 2 oz. to the greater weight, moments about the fulcrum give, if l ins. be the length of the lever,

$$10\left(\frac{l}{3}-\frac{4}{7}\right)=4\left(\frac{2l}{3}+\frac{4}{7}\right),$$

whence $$l=12 \text{ ins.}$$

10. Let AB and BC be the two levers; F_1 and F_2 be their fulcra respectively; P and Q be the pressures at the hinge, B, and R be the reaction of the press at C. Then if a force of 10 stone wt. be applied at A, the long end of the first lever, we have

$$10\times 3=P\times\frac{1}{2},\quad P=Q,$$

and $$Q\times 3=R\times\frac{1}{2},$$

whence $$R=6Q=6P=360 \text{ stone wt.}$$

11. [Cf. the figure, p. 175.] Let W lbs. be the weight of the lever acting at its middle point. The power of 12 lbs. wt. acts vertically upwards, and the weight of 3 lbs. downwards Taking moments about the fulcrum, we have

$$W\times\frac{21}{2}+3\times 21=12\times 7,$$

whence $$W=2 \text{ lbs.}$$

Also, let P lbs. wt. be the required power, acting vertically upwards, when the weight of 3 lbs. is increased to 4 lbs. Then, since $W=2$ lbs., we have

$$P\times 5=2\times\frac{21}{2}+4\times 21,$$

whence $$P=21 \text{ lbs. wt.}$$

12. [Take the figure, p. 177, with the forces of 13 lbs. wt. and 14 lbs. wt. acting at A and B respectively, and with the directions of the arrowheads reversed.] If R lbs. wt. be the required pressure, by Art. 27, we have

$$R=\sqrt{(13)^2+(14)^2+2\,.\,13\,.\,14\left(-\frac{5}{13}\right)}=15 \text{ lbs. wt.}$$

13. [Take the figure, p. 177, with AOB straight, the forces proportional to $\sqrt{3}+1$ and $\sqrt{3}-1$ acting at A and B respectively, and with the directions of the arrowheads reversed.] Let AOB be the lever, O being the fulcrum, and let the lines of action of the forces meet in C. Then, if R be the required pressure, we have

$$R=\sqrt{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}=2\sqrt{2}.$$

Also, by Lami's Theorem,

$$\frac{\sin ACO}{\sqrt{3}-1}=\frac{\sin BCO}{\sqrt{3}+1}=\frac{\sin ACB}{R}\left(=\frac{1}{2\sqrt{2}}\right),$$

whence $$\sin ACO=\frac{\sqrt{3}-1}{2\sqrt{2}},$$

and $$\sin BCO=\frac{\sqrt{3}+1}{2\sqrt{2}},$$

i.e. $$\angle ACO=15^\circ, \text{ and } \angle BCO=75^\circ;$$

hence the $\angle COB=45^\circ$, *i.e.* the pressure is $2\sqrt{2}$ acting at an angle of 45° to the lever.

14. Let ACB be the lever, C be the angular point, and the lengths of the arms AC and CB be $5l$ and l respectively. Also, let the lines of action of the weight of 10 lbs. and R lbs. wt. the required pressure meet the horizontal line through C in M and N respectively. The angle $ACM=45^\circ$. Then moments about C give

$$R \cdot CN=10 \cdot CM,$$

i.e. $$R \cdot l\cos 45^\circ=10 \cdot 5l\cos 45^\circ,$$

whence $$R=50 \text{ lbs. wt.}$$

15. Let ACB be the rod, C be the angular point, and AC be the longer arm. The angle ACB is 120°. The weights of the arms are proportional to their lengths, *i.e.* $2:1$, $2W$ and W say, and act at their middle points. Let the directions of $2W$ and W meet the horizontal line through C in M and N respectively, and let the angle MCA be θ. Then, taking moments about C, we have

$$2W \cdot CM=W \cdot CN,$$

i.e. $$2W \cdot \frac{AC}{2}\cos\theta=W \cdot \frac{CB}{2}\cos(60^\circ-\theta),$$

or $$4\cos\theta=\cos(60^\circ-\theta)=\cos 60^\circ\cos\theta+\sin 60^\circ\sin\theta.$$

$$\therefore\ 8\cos\theta=\cos\theta+\sqrt{3}\sin\theta,$$

so that $$7\cos\theta=\sqrt{3}\sin\theta, \text{ whence } \tan\theta=\frac{7}{\sqrt{3}},$$

i.e. $$\theta=\tan^{-1}\frac{7}{\sqrt{3}}.$$

16. Let AB be the bar, its weight 17 lbs. acting at C its middle point. Let D be the edge of the table, and W lbs. be the required weight at B. Since $BC=3\frac{3}{4}$ ft., therefore $CD=1\frac{1}{4}$ ft.; and moments about D give $W\times 2\frac{1}{2}=17\times 1\frac{1}{4}$, whence $W=8\frac{1}{2}$ lbs.

17. If P be the power acting at a point X, we have

$$P \, . \, AX = W \, . \, AB, \text{ so that } P = W\frac{AB}{AX}.$$

The strain at the hinge

$$= W \sim P = W\left(1 \sim \frac{AB}{AX}\right) = W\frac{BX}{AX}$$

[or it may be obtained at once by moments about X]; if this must not exceed $\frac{W}{2}$, then, in the extreme case,

$$\frac{BX}{AX} = \frac{1}{2}, \text{ so that } \frac{BX}{AX} = \frac{1}{1}, \text{ or } \frac{1}{3},$$

according as X is within or without AB. Hence X ranges over a distance $\frac{4}{3}AB$.

18. Let A be the end of the oar in the man's hand, B the rowlock and C the end in the water. Then

$$AC = 3 \, . \, AB, \text{ so that } BC = 2AB.$$

The forces exerted on the oar at A and C balance about B, so that

$$\text{force at } C \times BC = 56 \times AB.$$

$$\therefore \text{ force at } C = 56 \times \frac{AB}{BC} = 56 \times \tfrac{1}{2} = 28 \text{ lbs. wt.}$$

$\therefore$ total force exerted by the water on the ends of the oars (*i.e.* total propelling force) $= 8$ times the force at $C = 224$ lbs. wt.

19. If W lbs. be the required weight, moments about the hinge give

$$W \times \frac{7}{8} = 3\tfrac{1}{2} \times 5,$$

whence $$W = 20 \text{ lbs.}$$

20. Suppose the crowbar applied at the middle point of the edge in a plane through the centre of gravity of the cube. Let AB and CBD represent the base of the cube and the crowbar respectively, B being the point of contact; then C is the fulcrum, and if P be the required force vertically upwards (*i.e.* the least force) at D, Q be the reaction at B on the cube, and W be the weight of the cube, then moments about A for the cube give

$$\frac{1}{2}W = Q,$$

and moments about C for the fulcrum give

$$P \, . \, 4 = Q \, . \, \frac{1}{2}.$$

$$\therefore \; P = \frac{1}{8} \times \frac{W}{2} = \frac{1}{8} \text{ ton} = 2\tfrac{1}{2} \text{ cwt.}$$

21. Let CBA be the crowbar, B being the point in contact with the block, and C the point in contact with the ground; and let D be the point of the block on the ground. Let R be the action at B, and P be the force applied at A, both perpendicular to the crowbar; and let the line of action of the weight of the block ($=nP$), acting at O its centre, meet the ground in the point E. Then, since the crowbar is at an angle of 60° to the ground, the $\angle DBC=$ the $\angle BDC=30°$, and

$$DC=CB=\frac{a}{2}\sec 30°=\frac{a}{\sqrt{3}};$$

also
$$DE=DO\cos 75°=\frac{a}{\sqrt{2}}\cdot\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{a(\sqrt{3}-1)}{4}.$$

For the crowbar, moments about C give

$$P.CA=R.CB \qquad \text{(1)};$$

for the block, moments about D give

$$nP.DE=R.DB\sin 60°,$$

i.e.
$$nP.\frac{a(\sqrt{3}-1)}{4}=R.\frac{a\sqrt{3}}{2} \qquad \text{(2)}$$

From (1) and (2), by division, we have

$$CA\div\left[\frac{na(\sqrt{3}-1)}{4}\right]=\frac{a}{\sqrt{3}}\div\frac{a\sqrt{3}}{2};$$

$$\therefore\ CA=\frac{2}{3}\cdot\frac{na(\sqrt{3}-1)}{4}=\frac{n}{6}(\sqrt{3}-1)a.$$

EXAMPLES. XXIII. (Pages 186, 187.)

1. (1) $W=2^nP=2^4\times 20=320$ lbs.

(2) $P=\frac{W}{2^n}=\frac{112}{2^4}=7$ lbs. wt.

(3) $2^n=\frac{W}{P}=\frac{56}{7}=8=2^3$, so that $n=3$.

2. By the formula $2^nP=W+w(2^n-1)$, we have

(1) $2^4P=97+2^4-1$, whence $P=7$ lbs. wt.

(2) $W=2^3\times 7-\frac{3}{2}(2^3-1)=45\frac{1}{2}$ lbs.

(3) $2^5\times 31=775+w(2^5-1)$, whence $w=7$ lbs.

(4) $2^n\times 2=107+\frac{1}{3}(2^n-1)$.

Hence $5\times 2^n=320$, so that $2^n=64=2^6$,

i.e. $n=6$.

3. Here $2T_1 = W + 2$, $2T_2 = T_1 + 2 = \frac{W}{2} + 3$,

$$2T_3 = T_2 + 2 = \frac{W}{4} + \frac{7}{2}, \text{ and } 2P = 2T_4 = T_3 + 2 = \frac{W}{8} + \frac{15}{4};$$

$$\therefore\ 2 \times 20 = \frac{W}{8} + \frac{15}{4}, \text{ whence } W = 290 \text{ lbs.}$$

4. Here $2T_1 = 69 + 9 = 78$, $2T_2 = T_1 + 2 = 41$,

and $2P = 2T_3 = T_2 + 1 = 21\frac{1}{2}$, whence $P = 10\frac{3}{4}$ lbs. wt.

5. Here $2T_1 = 54 + 4 = 58$, $2T_2 = T_1 + 3 = 32$,

$$2T_3 = T_2 + 2 = 18, \text{ and } 2P = 2T_4 = T_3 + 1 = 10,$$

so that $P = 5$ lbs. wt.

6. Here we have $T_4 = P$,

$$T_3 = 2T_4 - w = 2P - w, \quad T_2 = 2T_3 - w = 4P - 3w,$$

and $T_1 = 2T_2 - w = 8P - 7w$;

the stress on the beam

$$= T_1 + T_2 + T_3 + T_4 = 15P - 11w.$$

7. Here $2T_1 = 28 + 4 = 32$, $2T_2 = T_1 + 2 = 18$,

and $2P = T_2 + 1 = 10$, so that $P = 5$ lbs. wt.

8. If the pulleys are weightless,

$$\frac{W}{P} = 2^n;$$

if their weight be taken into account, then

$$W = 2^n P - w_1 - 2w_2 - \ldots\ldots$$

i.e. W is less than before, and $\frac{W}{P}$ is less; hence the supposition that the pulleys are weightless brings out the mechanical advantage greater than it actually is.

9. If W be the weight supported, and w be the weight of each pulley, we have

$$7 = \frac{W}{8} + \frac{w}{8} + \frac{w}{4} + \frac{w}{2}, \text{ and } 4 = \frac{W}{16} + \frac{w}{16} + \frac{w}{8} + \frac{w}{4} + \frac{w}{2},$$

i.e. $W + 7w = 56$, and $W + 15w = 64$,

whence $w = 1$ lb., and $W = 49$ lbs.

Or thus: (1) $2T_1 = W + w$, $2T_2 = T_1 + w$, and

$$2P = 2 \times 7 = T_2 + w;$$

$$\therefore\ 28 = 2T_2 + 2w = T_1 + 3w;$$

$$\therefore\ 56 = 2T_1 + 6w = W + 7w \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$(2)\quad 2T_1 = W + w,\quad 2T_2 = T_1 + w,\quad 2T_3 = T_2 + w,\text{ and}$$
$$2P = 2 \times 4 = T_3 + w;$$
$$\therefore\ W + w = 2T_1 = 4T_2 - 2w = -2w + 8T_3 - 4w = 8(8 - w) - 6w,$$
$$\therefore\ W + 15w = 64 \qquad \text{(ii)}.$$

From (i) and (ii) we have $w = 1$ lb., and $W = 49$ lbs.

10. Here $T_4 = P$, $T_3 = 2T_4 - 4w = 2P - 4w$,

$$T_2 = 2T_3 - 3w = 4P - 11w,\text{ and } T_1 = 2T_2 - 2w = 8P - 24w;$$

also $$15w + w = 2T_1 = 16P - 48w,$$

whence $$P = 4w.$$

The stress on the beam

$$= T_1 + T_2 + T_3 + T_4 = 15P - 39w$$
$$= 60w - 39w = 21w.$$

Or thus: The stress on the beam

$$= 15w + w + 2w + 3w + 4w - P = 25w - 4w = 21w.$$

11. Starting with P, we have

$$T_4 = P,\quad T_3 > 2P,\quad T_2 > 4P,\text{ and } T_1 > 8P;$$

hence, by addition, the stress on the beam $> 15P$.

Again, starting with W, we have

$$T_1 < \frac{W}{2},\quad T_2 < \frac{T_1}{2},$$

i.e. $$T_2 < \frac{W}{4},\quad T_3 < \frac{W}{8},\text{ and } T_4 < \frac{W}{16};$$

hence, by addition, the stress on the beam $< \frac{15W}{16}$.

12. The weight of the man is supported by the sum of the tensions of the strings passing round the pulleys, together with the pulling force, P, at the end of the string in his hand,

i.e. by $2^4P + P$, *i.e.* by $17P$;

$$\therefore\ 17P = 12\text{ stone, and } P = 9\tfrac{15}{17}\text{ lbs. wt.}$$

13. Here the weight of the man is supported by

$$2^4P - 10(2^4 - 1) + P,\ i.e.\text{ by } 17P - 150;$$
$$\therefore\ 17P - 150 = 156\text{ lbs.},$$

whence $$P = 18\text{ lbs. wt.}$$

EXAMPLES. XXIV. (Pages 189, 190.)

1. By the formula $W+w=nP$, we have

$$24+w=6\times 5, \text{ whence } w=6 \text{ lbs.}$$

2. Here $18+w=5n$, and $22+w=6n$, whence $n=4$, and $w=2$ lbs.

3. Here $5+w=4n$, and $18+w=5n$, whence $n=13$, and $w=47$ lbs.; since there are 13 strings, there must be 6 pulleys.

4. Here $28+w=6n$, and $42+w=8n$, whence $n=7$, and $w=14$ lbs.

5. If P be the tension of the string, nP is the tension of the string supporting the basket, where n is the number of strings; and

$$nP+P=W; \text{ hence } P=\frac{W}{n+1}.$$

6. The number of strings at the lower block $=2\times 4=8$; hence $8P=3$ cwt. $=(3\times 112)$ lbs. wt., so that $P=42$ lbs. wt. $=3$ stone wt.; hence, since the man weighs 12 stone, the required pressure

$$=12-3=9 \text{ stone wt.}$$

7. There were 4 pulleys in each block, *i.e.* 8 strings at the lower block; $\therefore$ $8P=18$, *i.e.* $P=2\frac{1}{4}$ tons.

8. If W goes up a distance x, each of the n strings joining the lower to the upper block shortens by x and thus the point of application of P goes down nx. Thus the velocity-ratio $=\dfrac{nx}{x}=n$.

Also since $W=nP$, $\therefore$ $W.x=P.nx$, *i.e.* the Principle of Work is true.

9. Here $n=4$; $\therefore$ $P=\dfrac{W}{4}=75$ lbs.

In the second case, if P_1 be the force, then only $\cdot 45P_1$ is usefully employed and $\therefore$ $\cdot 45P_1=\dfrac{W}{4}=75$. $\therefore$ $P_1=\dfrac{100}{45}\times 75=166\frac{2}{3}$.

10. Here $\dfrac{55}{100}P=\dfrac{1}{8}\times 5$ cwt. $\therefore$ $P=\dfrac{100}{88}=\dfrac{25}{22}=1\frac{3}{22}$ cwt.

EXAMPLES. XXV. (Pages 195, 196.)

1. By the formula $W=P(2^n-1)$, we have

(1) $W=2(2^4-1)=30$ lbs.

(2) $124=P(2^5-1)=31P$, whence $P=4$ lbs. wt.

(3) $105=7(2^n-1)$, so that $7\times 2^n=112$.

$$\therefore\ 2^n=16=2^4, \text{ i.e. } n=4.$$

2. By the formula $W=(2^n-1)P+w[2^n-n-1]$, we have

(1) $W=(2^4-1)10+1[2^4-4-1]=161$ lbs.

(2) $114=(2^3-1)P+\frac{1}{2}[2^3-3-1]=7P+2,$

whence $$P=16 \text{ lbs. wt.}$$

(3) $106=(2^5-1)3+w[2^5-5-1]=93+26w,$

whence $$w=\frac{1}{2}\text{ lb.}$$

(4) $137=(2^n-1)4+\frac{1}{2}[2^n-n-1].$

$$\therefore\ 274=8\cdot2^n-8+2^n-n-1,$$

so that $$9\cdot2^n-n=283=288-5=9\times32-5=9\cdot2^5-5,$$

$$i.e.\ n=5.$$

3. We have

$$336=(2^5-1)P+1[2^5-5-1]=31P+26,$$

whence $$P=10 \text{ lbs. wt.}$$

Again, take the figure as in Art. 150, with 5 pulleys, and T_5 meeting the bar in A; let

$$a=AD=DE=EF=FG,$$

and X be the required point; we have

$$T_1=10,\quad T_2=2T_1+1=21,\quad T_3=2T_2+1=43,$$
$$T_4=2T_3+1=87, \text{ and } T_5=2T_4+1=175;$$

$$\therefore\ AX=\frac{175\times0+87\times a+43\times2a+21\times3a+10\times4a}{175+87+43+21+10}$$

$$=\frac{276a}{336}=\frac{23a}{28};$$

$$\therefore\ AX:XD=23:5.$$

4. Here $T_1=P$, $T_2=2T_1=2P$, $T_3=2T_2=4P$, and $T_4=2T_3=8P$;

$$\therefore\ DX=\frac{8P\times0+4P\times1+2P\times2+P\times3}{8P+4P+2P+P}=\frac{11}{15}\text{ inch.}$$

5. Here $$T_1=P,\ T_2=2T_1+w=2P+\frac{W}{64},$$

$$T_3=2T_2+w=4P+\frac{3W}{64},$$

$$T_4=2T_3+w=8P+\frac{7W}{64},$$

and $$W=T_1+T_2+T_3+T_4=15P+\frac{11W}{64};$$

$$\therefore\ 53W=960P,\ i.e.\ \frac{W}{P}=\frac{960}{53}=18\tfrac{6}{53}.$$

6. Take the figure as in Art. 150 with 3 pulleys only,

$$DE=1 \text{ inch}=EF;$$

and we have

$$DX=\frac{T_3\times 0+T_2\times 1+T_1\times 2}{T_3+T_2+T_1}=\frac{2P+2P}{2^2P+2P+P}=\frac{4}{7}\text{ inch.}$$

7. Here $\quad T_1=P=w,\quad T_2=2T_1+w=3w,$

$$T_3=2T_2+w=7w,\quad T_4=2T_3+w=15w,$$

and $\quad T_5=2T_4+w=31w;$

$$\therefore\ W=T_1+T_2+T_3+T_4+T_5=57w=57P.$$

8. When $n=3$, the formula $W=(2^n-1)P+w[2^n-n-1]$
gives

$$W=7P+4w;$$

if $P=0$, and $w=2$ oz., then $W=8$ oz. $=\frac{1}{2}$ lb.;

if $P=3$ lbs. wt., and $W=25$ lbs., then $w=1$ lb.

9. By the formula $W=P(2^n-1)$, we have

$$W=70(2^n-1),\text{ and }W=150(2^{n-1}-1),$$

so that, by division,

$$\frac{2^n-1}{2^{n-1}-1}=\frac{15}{7}=\frac{16-1}{8-1}=\frac{2^4-1}{2^3-1}.$$

$$\therefore\ n=4,\text{ and }W=70(2^4-1)=1050\text{ lbs.}$$

10. In the usual case,

$$W=T_1+T_2+\ldots+T_n=T_1(2^n-1);$$

if the string be attached to the weight, then

$$W=2T_1+T_2+\ldots+T_n=T_1+T_1(2^n-1)=T_1\cdot 2^n;$$

hence T_1 is diminished in the ratio of $2^n-1:2^n$, and this result depends on the value of n. If

$$2^n-1\cdot 2^n=15:16,$$

i.e.

$$\frac{2^n-1}{2^n}=\frac{15}{16},$$

then

$$1-\frac{1}{2^n}=1-\frac{1}{16}=1-\frac{1}{2^4},\text{ and hence }n=4.$$

11. Here, in the formula

$$W=(2^n-1)P+w[2^n-n-1]$$

we have

$$nw=W';$$

hence

$$W=(2^n-1)P+w(2^n-1)-W',$$

i.e.

$$W+W'=(2^n-1)(P+w);$$

hence $P+w$ will support the weight $W+W'$ if the pulleys had no weight.

[Cf. the formula $W=P(2^n-1)$.]

12. W' is supported by two portions of string. The tension of the attached string being uniform is equal to $\frac{W'}{2}$, so that W and P are practically increased each by $\frac{W'}{2}$; hence the formula $W=P(2^n-1)$ becomes

$$W+\frac{W'}{2}=\left(P+\frac{W'}{2}\right)(2^n-1),$$

i.e. $$W=P(2^n-1)+W'.2^{n-1}-\frac{W'}{2}-\frac{W'}{2},$$

or $$W=P(2^n-1)+W'(2^{n-1}-1).$$

13. In the figure of Art. 150, let the line of action of P meet $DEFG$ in the point O.

$$T_1=P,\quad T_2=2P,\quad T_3=4P,\ \&c.$$

$$OX=\frac{P\times 2a+2P\times 3a+4P\times 4a+8P\times 5a+\dots}{P+2P+4P+8P+\dots}$$

$$=\frac{2.1+3.2+4.2^2+5.2^3+\dots\text{ to } n \text{ terms}}{1+2+4+8+\dots\text{ to } n \text{ terms}}\times a.$$

[If $$S=2.1+3.2+4.2^2+5.2^3+\dots+(n+1)\,2^{n-1},$$

then $$2S=\quad 2.2+3.2^2+4.2^3+\dots+n.2^{n-1}+(n+1)\,2^n$$

$$\therefore\ -S=2.1+1.2+1.2^2+1.2^3+\dots+1.2^{n-1}-(n+1)\,2^n$$

$$\therefore\ S=(n+1)\,2^n-2-[2+2^2+2^3+\dots\text{ to } (n-1)\text{ terms}]$$

$$=(n+1)\,2^n-2-2\,(2^{n-1}-1)=n.2^n;$$

hence $$OX=\frac{n.2^n}{2^n-1}a=\frac{2^n}{2^n-1}na.$$

EXAMPLES. XXVI. (Pages 201–203.)

1. Take the figure p. 198, with

$$BC=3\text{ ft., } AB=4\text{ ft., and } W=16\text{ lbs.}$$

Then $$AC=\sqrt{3^2+4^2}=5\text{ ft.};$$

hence the ratios of α are known;

also $$P=W\tan\alpha=16\times\frac{3}{4}=12\text{ lbs. wt.},$$

and $$R=W\sec\alpha=16\times\frac{5}{4}=20\text{ lbs. wt.}$$

2. Take the figure p. 197. We have

$$P = W \sin \alpha, \text{ i.e. } \frac{W}{2} = W \sin \alpha.$$

$$\therefore \sin \alpha = \frac{1}{2}, \text{ i.e. } \alpha = 30^\circ;$$

and $$R = W \cos \alpha = W \frac{\sqrt{3}}{2}.$$

3. Take the figure p. 199. Let P lbs. wt. be the required tension, then R and P are equally inclined to the vertical direction of the weight of 180 lbs., and are, therefore, obviously equal. Hence resolving vertically, we have $2P \cos 30^\circ = 180$,

whence $$P = \frac{180}{\sqrt{3}} = 103{\cdot}92 \text{ lbs. wt.}$$

This is the greatest tension the rope could exert, since it is stated in the question to be at the point of breaking.

4. The force P, the reaction of the plane R, and the weight of the body W act in directions making angles of 120° with one another; hence $P = R = W$.

5. Let α = the inclination of the plane to the horizon = the angle between the forces P. Then resolving along the plane, we have

$$P + P \cos \alpha = 2P \sin \alpha.$$

$$\therefore 1 + \cos \alpha = 2 \sin \alpha,$$

so that $$2 \cos^2 \frac{\alpha}{2} = 4 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}, \text{ i.e. } \cot \frac{\alpha}{2} = 2;$$

$$\therefore \frac{\text{base}}{\text{height}} = \cot \alpha = \frac{\cot^2 \frac{\alpha}{2} - 1}{2 \cot \frac{\alpha}{2}} = \frac{4-1}{4} = \frac{3}{4}.$$

Again, resolving perpendicular to the plane, we have

$$R = P \sin \alpha + 2P \cos \alpha$$

$$= P \times \frac{4}{5} + 2P \times \frac{3}{5} = 2P.$$

6. Take the figure p. 199, with $\alpha = 30^\circ = \theta$, and we have

$$P \cos 30^\circ = W \sin 30^\circ, \text{ i.e. } \frac{W}{P} = \frac{\cos 30^\circ}{\sin 30^\circ} = \frac{\sqrt{3}}{1}.$$

7. Take the figure p. 199. We have given

$$W : P : R = 4 : 3 : 2.$$

Since P = the resultant of R and W, we have

$$P^2 = R^2 + 2R \cdot W \cos(180° - a) + W^2,$$

i.e. $$9 = 4 + 2 \cdot 2 \cdot 4 \cos(180° - a) + 16,$$

whence $$16 \cos a = 11, \text{ i.e. } a = \cos^{-1}\frac{11}{16}.$$

Resolving perpendicular to the plane, we have

$$R + P \sin\theta = W \cos a,$$

i.e. $$2 + 3 \sin\theta = 4 \times \frac{11}{16},$$

whence $$\sin\theta = \frac{1}{4}, \text{ i.e. } \theta = \sin^{-1}\frac{1}{4}.$$

8. Take the figure p. 199, with $a = 30° = \theta$, $W = 5$ lbs., and an additional force of 2 lbs. wt. acting parallel to the plane upwards. Then resolving along and perpendicular to the plane, we have

$$2 + P \cos 30° = 5 \sin 30°, \text{ and } R + P \sin 30° = 5 \cos 30°;$$

i.e. $$2 + \frac{P\sqrt{3}}{2} = \frac{5}{2}, \text{ and } R + \frac{P}{2} = \frac{5\sqrt{3}}{2};$$

whence $$P = \frac{1}{\sqrt{3}} \text{ lb. wt., and } R = \frac{7}{\sqrt{3}} \text{ lbs. wt.}$$

9. Take the figure p. 197, with $W = 10$ lbs. Then we have

$$P = 10 \sin a, \text{ and } R = 10 \cos a;$$

but $P + 10 = 2R$, since P, R and 10 are in arithmetical progression;

$$\therefore 10 \sin a + 10 = 20 \cos a, \text{ or } 1 + \sin a = 2 \cos a.$$

$$\therefore (1 + \sin a)^2 = 4(1 - \sin^2 a), \text{ i.e. } 1 + \sin a = 4(1 - \sin a),$$

whence $$\sin a = \frac{3}{5}; \text{ hence } P = 10 \times \frac{3}{5} = 6 \text{ lbs. wt.}$$

10. Taking the figure p. 197, we have

$$P = W \sin a \quad \text{.........................(i)},$$

and $$R = W \cos a \quad \text{.........................(ii)}.$$

Again, taking the figure p. 198, with R' for R, and R for W, we have

$$R' = P \sin a + R \cos a,$$

i.e. by (i) and (ii), $$R' = W \sin^2 a + W \cos^2 a$$

$$= W(\sin^2 a + \cos^2 a) = W.$$

11. Let α and β be the inclinations to the horizon of the planes on which rest the bodies of weight W and 12 lbs. respectively, and let T be the tension of the string. Then, resolving along each plane, we have

$$T=W\sin\alpha, \text{ and } T=12\sin\beta,$$

so that
$$W\sin\alpha=12\sin\beta;$$

but, if h be the height of the vertex, we have

$$\sin\alpha=\frac{h}{11}, \text{ and } \sin\beta=\frac{h}{8};$$

$$\therefore\ W\times\frac{h}{11}=12\times\frac{h}{8}, \text{ whence } W=16\tfrac{1}{2}\text{ lbs.}$$

12. If there be $2n$ trucks in all, W tons be the weight of a truck, and T be the tension of the connecting chain, then resolving along the two parts of the tramway, we have

$$T=n(W+1)\sin\alpha, \text{ and } T=nW\sin\beta;$$

whence
$$W=\frac{\sin\alpha}{\sin\beta-\sin\alpha}\text{ tons.}$$

13. Take the figure p. 199, with $R=P$, and produce ED vertically upwards to any point S. Since $W=$ the resultant of two equal forces P, its direction must bisect the angle between them. Hence the $\angle PDS=$ the $\angle RDS=\alpha$; therefore the $\angle PDC=\frac{\pi}{2}-2\alpha$.

14. Let x and y be the portions of the string on the plane (whose inclination to the horizon is α) and hanging vertically respectively. Their weights are proportional to their lengths, Wx and Wy, say. Then, if T be the tension of the pulley, we have

$$T=Wx\sin\alpha, \text{ and } T=Wy;$$

$$\therefore\ x\sin\alpha=y, \text{ i.e. } x:y=1:\sin\alpha.$$

15. Let BAC and BDC be the two planes, so that BC is the common height; and let $AB=2BC$, and $BD=2CD$; then the $\angle BAC=30°$, and the $\angle BDC=60°$. Let W and W' be the weights on BA and BD, and R and R' be the pressures respectively. Then, if T be the tension of the string, resolving along and perpendicular to each plane, we have

$$T=W\sin 30°, \text{ and } T=W'\sin 60°;$$

$$R=W\cos 30°, \text{ and } R'=W'\cos 60°;$$

whence
$$R\tan 30°=R'\tan 60°,$$

i.e.
$$\frac{R}{\sqrt{3}}=R'\sqrt{3}, \text{ so that } R=3R'.$$

16. Take the figure p. 197, with $a=20°\,20'$, and $W=50$ lbs. Then we have

$$P=50\sin 20°\,20',\ \text{ and }\ R=50\cos 20°\,20';$$

whence $\quad P=50\times\cdot3474812=17\cdot37406$ lbs. wt.,

and $\quad R=50\times\cdot9376869=46\cdot88435$ lbs. wt.

17. Take the figure p. 199, with $a=25°$, $\theta=35°$, and $W=20$ lbs. By Lami's Theorem, we have

$$\frac{P}{\sin 25°}=\frac{20}{\sin 55°}=\frac{R}{\sin 150°},\ \textit{i.e.}\ P=\frac{20\sin 25°}{\sin 55°},$$

$$\begin{aligned}\therefore\ \log P&=\log 20+L\sin 25°-L\sin 55°\\&=1\cdot3010300+9\cdot6259483-9\cdot9133645\\&=1\cdot0136138;\end{aligned}$$

whence $\quad P=10\cdot318$ lbs. wt.

Also $\quad R=\dfrac{20\sin 150°}{\sin 55°}=\dfrac{10}{\sin 55°};$

$$\begin{aligned}\therefore\ \log R&=\log 10-L\sin 55°+10=11-9\cdot9133645\\&=1\cdot0866355;\end{aligned}$$

whence $\quad R=12\cdot208$ lbs. wt.

18. Take the figure p. 198, with $a=28°\,15'$, and $W=30$ lbs. Then we have

$$P=30\tan 28°\,15',\ \text{ and }\ R=30\sec 28°\,15';$$

hence $\quad P=30\times\cdot5373194=16\cdot119582$ lbs. wt.,

and $\quad Q=30\times1\cdot1352146=34\cdot056438$ lbs. wt.

EXAMPLES. XXVII. (Pages 208, 209.)

1. Here $\quad P\times24=56\times3$, whence $P=7$ lbs. wt.

2. Here $\quad W\times5=20\times30$, whence $W=120$ lbs.

The pressure $\quad =P+W=20+120=140$ lbs. wt.

Again, if the thickness of the ropes be each 1 inch, we have

$$W\left(5+\frac{1}{2}\right)=20\left(30+\frac{1}{2}\right),\ \text{ whence }\ W=110\tfrac{10}{11}\text{ lbs.}$$

3. If R ins. be the required radius, we have

$$3\times R=30\times2,\ \text{ whence }\ R=20\text{ ins.}$$

4. Let x feet be the required distance; then, since the radius of the axle $=\dfrac{8}{12}$ foot, *i.e.* $\dfrac{2}{3}$ foot, we have

$$8\times26\tfrac{2}{3}\times x=2240\times\frac{2}{3},\ \text{ whence }\ x=7\text{ feet.}$$

5. The point of application is 6 feet 4 inches, *i.e.* $6\frac{1}{3}$ feet, from the axis, and we have

$$4 \times 112 \times 6\tfrac{1}{3} = W \times \frac{1}{3}, \text{ whence } W = 3\tfrac{4}{5} \text{ tons.}$$

6. If the wheels and axles in contact be connected by strings tied to each, P be the power required, and T_1, T_2 and T_3 be the tensions of the successive strings, then we have

$$5P = T_3, \quad 5T_3 = T_2, \quad 5T_2 = T_1, \text{ and } 5T_1 = 1875 \text{ lbs};$$

hence $$P = \frac{1875}{5^4} = 3 \text{ lbs. wt.}$$

[The wheels and axles may act by means of cogs, when T_1, T_2 and T_3 above would be the reactions.]

7. Let A and B be the points from which the strings hang on the wheel and the axle, T_1 and T_2 be the tensions of the strings respectively, and X lbs. be the weight to be hung from B. Then

$$AB = 2 \text{ feet } 2 \text{ ins.},$$

and $$24T_1 = 2T_2, \text{ i.e. } 12T_1 = T_2 \qquad (1)$$

Also, for the equilibrium of the rod, we have

$$T_1 + T_2 = 10 + X \qquad (2),$$

and $$T_1 = T_2 - X \qquad (3).$$

Using (1) in (2) and (3), we have

$$13T_2 = 10 + X, \text{ and } 11T_2 = X;$$

$$\therefore 13P = 11(10 + X), \text{ whence } X = 55 \text{ lbs.}$$

8. With the figure of Art 164, treating the wheel and-axle as the two parts of the axle therein, and the arm 2 feet in length answering to the wheel in Art. 164, we have

$$P \times 24 = \frac{112}{2}(12 - 2), \text{ whence } P = 23\tfrac{1}{3} \text{ lbs wt.}$$

9. Here $$P = \frac{W}{2} \cdot \frac{c-a}{b} = 28 \times \frac{5-4}{12} = 2\tfrac{1}{3} \text{ lbs. wt.}$$

10. Here $$20 = \frac{W}{2} \cdot \frac{6-4}{18}; \text{ whence } W = 360 \text{ lbs}$$

11. If the weights be fastened at the points A and B, C be the centre of the wheel, and G be the centre of gravity of the two weights, then CG bisects the angle ACB; therefore the angle $ACG = 60°$, and $CG = \frac{1}{2} CA = 6$ ins. Also the moment about C of the 20 lbs at G is greatest when CG is horizontal, and then, if W lbs. be the required weight, we have

$$W \times 1 = 20 \times 6, \text{ i.e. } W = 120 \text{ lbs.}$$

Otherwise thus: Let one weight be at an angle θ from the horizontal, and therefore the other at an angle $(120^\circ - \theta)$. Then we have

$$W \times 1 = 10 \times 12 \cos\theta + 10 \times 12 \cos(120^\circ - \theta),$$
$$\therefore\ W = 120\,[\cos\theta + \cos(120^\circ - \theta)]$$
$$= 120 \times 2 \cos 60^\circ \cos(60^\circ - \theta) = 120 \cos(60^\circ - \theta);$$

the greatest value of $\cos(60^\circ - \theta)$ is unity, so that $W = 120$ lbs.

12. The weight of the body raised

$$= 5 \times 6 \text{ lbs.} = 30 \text{ lbs.};$$

hence the work expended

$$= (30 \times 50) \text{ ft.-lbs.} = 1500 \text{ ft.-lbs.}$$

13. If $$\alpha = \cos^{-1}\frac{4}{5},$$

then $$\cos\alpha = \frac{4}{5};$$

therefore $$\sin\alpha = \frac{3}{5}.$$

The mechanical advantage of the capstan

$$= \frac{2\pi \times 60}{2\pi \times 10} = 6.$$

The work done in raising 1 ton through $\left(35 \times \frac{3}{5}\right)$ feet, *i.e.* 21 feet, vertically

$$= (2240 \times 21) \text{ ft.-lbs.} = 47040 \text{ ft.-lbs.}$$

The force at the end of the lever

$$= \left(\frac{1}{6} \times \frac{3}{5} \times 20\right) \text{ cwt.} = 2 \text{ cwt.};$$

also it must act through (6×35) feet, *i.e.* 210 feet.

14. In Art. 164, if the machine turns through an angle θ portions $c\theta$, $a\theta$ are wound on and unwound; hence the pulley goes up $\frac{c-a}{2}\theta$; also $b\theta$ is unwound so that P goes down $b\theta$. Hence the Principle of Work is true if $P \cdot b\theta = W\frac{c-a}{2}\theta$, and this is the result of Art. 164. Also the velocity-ratio is $b \div \frac{c-a}{2}$, *i.e.* $\frac{2b}{c-a}$. The same argument holds for Art. 165.

EXAMPLES. XXVIII. (Pages 216, 217.)

1. Let W_1 and W_2 be the weights of the scale-pans, and W lbs. be the true weight of the body. Then, since the arms are of equal length, we have

$$W+W_1=10+W_2, \text{ and } 12+W_1=W+W_2;$$

hence, by subtraction,

$$W-12=10-W, \text{ and } W=11 \text{ lbs.}$$

2. Let the real weight be W lbs.; then we have

$$W\times 9=8\tfrac{3}{4}\times 27,$$

whence

$$W=26\tfrac{1}{4} \text{ lbs.}$$

3. Let W oz. be the required weight. Then, since the arms are of equal length, we have

$$18+W=20,$$

so that

$$W=2 \text{ oz.}$$

4. Let a and b be the lengths of the arms, and W lbs. be the true weight; then we have

$$Wa=9b, \text{ and } 4a=Wb;$$

hence, by multiplication, $4Wa^2=9Wb^2$;

$$\therefore 2a=3b, \text{ i.e. } a:b=3:2.$$

Also

$$W=9\frac{b}{a}=6 \text{ lbs.}$$

5. As in Ex. 2, p. 215, the true weight

$$=\sqrt{24\times 25}=5\sqrt{24}=5\times 4{\cdot}8989=24{\cdot}494 \text{ lbs.}$$

6. Let a and b be the lengths of the arms for the pans A and B respectively, and W be the true weight of the piece of lead. Then we have

$$Wa=100b, \text{ and } Wb=104a;$$

hence, by division, $\frac{a}{b}=\frac{100b}{104a}$, i.e. $\frac{a^2}{b^2}=\frac{100}{104}=\frac{25}{26}$;

$$\therefore \frac{a}{b}=\frac{5}{\sqrt{26}}, \text{ i.e. } a:b=5:\sqrt{26}.$$

7. Let x ins. be the length of the longer arm, and W lbs. be the true weight of the body. Then we have

$$10 \times x = W \times 12, \text{ and } W \times x = 11 \times 12.$$

Hence, by division, $\frac{10}{W} = \frac{W}{11}$, so that $W = \sqrt{110}$ lbs.

Also, $$x = \frac{12W}{10} = \frac{6}{5}\sqrt{110} \text{ ins.}$$

8. As in Ex. 2, p. 215, if the seller appear to weigh out quantities equal to W from each end, he really gives his customer

$$\frac{10}{9}W + \frac{9}{10}W,$$

i.e. $$\frac{181}{90}W, \; i.e. \; 2W + \frac{1}{90}W;$$

hence he loses $\frac{1}{90}W$ on $2W$, *i.e.* he loses

$$\left[\left(\frac{1}{90}\right) \div 2\right] \times 100 \text{ per cent., } i.e. \; \frac{5}{9}\text{th per cent.}$$

9. (1) Let W lb. be the real weight of the quantity which appears to weigh 1 lb.; then

$$W \times 9 = 8 \times 1, \; i.e. \; W = \frac{8}{9} \text{ lb.}$$

Hence the customer receives $\frac{8}{9}$ lb. for 2*s.*; therefore he pays at the rate of $\left(\frac{9}{8} \times 2s.\right)$ per lb., *i.e.* 2*s.* 3*d.* per lb.

(2) Here, what appears to weigh 1 lb. really weighs $\frac{9}{8}$ lb.; therefore the price is really only $\left(\frac{8}{9} \times 2s.\right)$ per lb., *i.e.* 1*s.* 9⅓*d.* per lb.

10. The arms of the balance are in the ratio of 19 : 20. From the shorter arm he really weighs out $\left(\frac{20}{19} \times 9\frac{1}{2}\right)$ lbs., *i.e.* $\left(\frac{1}{19} \times 9\frac{1}{2}\right)$ lb. too much; and from the longer arm he weighs out $\left(\frac{19}{20} \times 9\frac{1}{2}\right)$ lbs., *i.e.* $\left(\frac{1}{20} \times 9\frac{1}{2}\right)$ lb. too little. Hence, on the whole he sells too much by

$$9\tfrac{1}{2}\left(\frac{1}{19} - \frac{1}{20}\right) \text{ lb., } i.e. \text{ by } \frac{1}{40} \text{ lb.,}$$

and, therefore, he loses $\left(\frac{1}{40} \times 40\right)$ *s.*, *i.e.* 1 shilling.

11. If the lengths of the arms be a and b, and the corresponding scale-pans weigh x grains and y grains, then we have

$$(x+51\cdot075)\,a=(y+51\cdot362)\,b,$$

and $$(x+25\cdot592)\,a=(y+25\cdot879)\,b.$$

Therefore, by subtraction, $a\,(25\cdot483)=b\,(25\cdot483)$,

i.e. the arms are of equal length. Hence, we have

$$x-y=51\cdot362-51\cdot075=\cdot287 \text{ grains.}$$

12. Let a and b be the lengths of the arms; then we have

$$P\times a=Q\times b, \text{ and } \frac{101}{100}Q\times a=P\times b.$$

Hence, by division, $$\frac{100}{101}\cdot\frac{P}{Q}=\frac{Q}{P};$$

$$\therefore \left(\frac{P}{Q}\right)^2=\frac{101}{100}=1+\frac{1}{100},$$

$$\therefore \frac{P}{Q}=\left(1+\frac{1}{100}\right)^{\frac{1}{2}}=1+\frac{1}{200}, \text{ nearly}=1\cdot005, \text{ nearly};$$

hence $$\frac{a}{b}=\frac{Q}{P}=\left(1+\frac{1}{100}\right)^{-\frac{1}{2}}=1-\frac{1}{200}, \text{ nearly}=\cdot995, \text{ nearly.}$$

13. If R be the load, and W be the true weight, we have

$$W+R=P, \text{ and } Q+R=W;$$

hence, by addition, $Q+2R=P$, *i.e.* $R=\dfrac{P-Q}{2}$;

and, by subtraction,

$$W-Q=P-W, \text{ i.e. } W=\frac{P+Q}{2}.$$

14. Let a and b be the lengths of the arms, and x be the required weight; then we have

$$P\,.\,a=(w+x)\,b, \text{ and } w'\,.\,a=(P+x)\,b;$$

hence, by division, $$\frac{P}{w'}=\frac{w+x}{P+x},$$

whence $$x=\frac{ww'-P^2}{P-w'};$$

and, by subtraction,

$$(P-w')\,a=(w-P)\,b.$$

$$\therefore \frac{a}{b}=\frac{w-P}{P-w'}.$$

15. Let the load be equivalent to S at the end of the arm a, b be the length of the other arm, and the weight to be weighed be each time weighed from the end of the arm a, then, if X be the apparent weight of R, we have

$$(P+S)a=Qb,\quad (Q+S)a=Rb,\quad (R+S)a=Xb;$$
$$\therefore (P-Q)a=(Q-R)b, \text{ and } (Q-R)a=(R-X)b;$$
$$\therefore (P-Q)(R-X)=(Q-R)^2,$$

i.e.
$$X=R-\frac{(Q-R)^2}{P-Q}.$$

16. Let A and B be the middle points of the ends $\frac{1}{2}$ inch and $\frac{1}{4}$ inch thick respectively; then, if G be the centre of gravity, we have, by Art. 118,

$$AG : GB=2\times\frac{1}{4}+\frac{1}{2} : 2\times\frac{1}{2}+\frac{1}{4}=1 : \frac{5}{4}=4 : 5;$$

but $20 . AG=W . GB$, if W lbs. be the true weight.

$$W=\frac{20\times 4}{5}=16 \text{ lbs.}$$

17. Here we have, if the load be W' at a distance c along the arm whose length is a,

$$W . a+W'c=P . b, \text{ and } Q . a+W'c=W . b.$$

Hence, by subtraction, $(W-Q)a=(P-W)b$.

$$\therefore \frac{a}{b}=\frac{P-W}{W-Q}.$$

18. Let a be the length of the arm of the balance, x be the distance from the fulcrum of the point at which he pushes, and R be the vertical component of the force he exerts.

As far as the beam is concerned, the effect of his pushing is to cause two forces to act on the beam, one R upwards at a distance x, and the other R downwards at a distance a.

The sum of the moments of these$=R . a-R . x=R(a-x)=$a positive moment tending to lower the mass. This moment must be overcome by additional weights in the other scale-pan, *i.e.* he appears to weigh more than before

EXAMPLES. XXIX. (Pages 222—224.)

1. Take the figure, p. 218. If C be the fulcrum, the 1 cwt. be suspended at A, G be the centre of gravity of the machine at a distance of 8 ins. to the right of C, and the movable weight of 12 lbs. be at X, then moments about C give

$$112\times 4=10\times 8+12CX, \text{ whence } CX=34\tfrac{2}{3} \text{ ins.}$$

2. Let AB be the rod, G be its centre of gravity where its weight 3 lbs. acts, and C be the fulcrum. Let a weight of 12 lbs. be placed at A; the movable weight of 2 lbs. must then be placed at B. If

$$AC = x \text{ ins., then } BC = (14\tfrac{1}{2} - x) \text{ ins.;}$$

also

$$AG = 1\tfrac{2}{3} \text{ in.}$$

Taking moments about C, we have

$$12x + 3(x - 1\tfrac{2}{3}) = 2(14\tfrac{1}{2} - x),$$

whence

$$x = 2 \text{ ins.}$$

The graduations for 2 lbs. are at distances equal to AC, *i.e.* 2 ins.; hence the graduations for 1 lb. are 1 in. apart.

3. Let W' be the weight of the machine, a be the distance of the centre of gravity from the fulcrum, and x ins. be the required distance. Then

$$15 \times 1 = W' \times a + \frac{6}{16} \times 8 \quad \text{......(i)},$$

and

$$24 \times 1 = W' \times a + \frac{6}{16} \times x \quad \text{......(ii)};$$

hence, by subtraction, $9 = \frac{6}{16}(x - 8)$, whence $x = 32$ ins.

4. Take the figure, p. 218, with C between A and G, 3 lbs. for W', and 2 lbs. for P. Then $AC = 1\tfrac{1}{3}$ in., and $CG = 2$ ins.; and we have

$$2CX + 3 \times 2 = W \times \frac{4}{3},$$

$$\therefore CX = \frac{2W}{3} - 3;$$

the least value W can have is $4\tfrac{1}{2}$ lbs., in which case $CX = 0$.

If $W = 5$ lbs., $CX = \frac{1}{3}$ in.; if $W = 6$ lbs., $CX = 1$ in., &c., the distances increasing by $\frac{2}{3}$ in.

5. Take the figure, p. 218, with C between A and G, 15 lbs. for W, 4 lbs. for W', and 3 lbs. for P. Then $AC = 8$ ins., and $CG = 3$ ins.; and we have

$$15 \times 8 = 4 \times 3 + 3CX,$$

whence

$$CX = 36 \text{ ins.};$$

but

$$CB = (48 - 8) = 40 \text{ ins.},$$

so that

$$BX = 4 \text{ ins.}$$

6. The weight of the bar acts at its middle point. Let C be the point of support, so that AC is 4 ins., and let a weight W lbs. be suspended at A. W is greatest when the movable weight is at B. Moments about C give

$$W \times 4 = 3 \times 8 + 2 \times 20,$$

whence $$W = 16 \text{ lbs.}$$

Now remove W, and let O be the point from which the graduations are measured; then

$$2CO + 3 \times 8 = 0, \text{ i.e. } CO = -12 \text{ ins.};$$

hence the required point is 12 inches to the left of C, *i.e.* is 8 inches to the left of A, the point at which the weight is attached.

7. Let AB be the rod, and W lb. be its weight acting at its middle point; let P lb. be the movable weight, and a be the length of each of the equal parts. When weighing the greatest weight, suspended at A, P must be at B, and we have

$$P \times 19a + W \times 9a = 20a \ldots\ldots\ldots\ldots \text{(i)}.$$

When weighing the least weight, suspended at A, P must be removed, and we have

$$W \times 9a = 2a,$$

whence $W = \frac{2}{9}$ lb. Hence, from (i), $P = \frac{18}{19}$ lb.

8. The weight of the rod, AB, acts at its middle point. Let C be the fulcrum, so that $AC = 2$ ins. If W lbs. be the greatest weight, suspended at A, the sliding weight must be at B, and we have

$$W \times 2 = 3 \times 10 + 1 \times 22,$$

whence $$W = 26 \text{ lbs.}$$

If W' lbs. be the least weight, suspended at A, the sliding weight must now be at C, and we have

$$W' \times 2 = 3 \times 10, \text{ whence } W' = 15 \text{ lbs.};$$

Again, if X be the position of the sliding weight to shew 20 lbs., we have

$$1 \times CX + 3 \times 10 = 20 \times 2, \text{ whence } CX = 10 \text{ ins.};$$

i.e. X is at the middle point of the rod.

9. Take the figure, p. 218, with C between A and G, so that

$$AC = 4 \text{ ins., and } CG = 1\tfrac{1}{2} \text{ in.};$$
$$W = 24 \text{ lbs., and } W' = 6 \text{ lbs.}$$

The movable weight P is at B. Then, moments about C give

$$P \times 29 + 6 \times \frac{3}{2} = 24 \times 4, \text{ whence } P = 3 \text{ lbs.}$$

10. Let AB be the bar, G be its middle point where its weight $2\frac{1}{4}$ lbs. acts, and C be the fulcrum, so that $AC=4$ ins. The greatest weight, W lbs. (suspended at A), is found when the movable weight is at B, and then we have

$$W\times 4=2\tfrac{1}{4}\times 14+1\times 32,\quad \text{whence}\quad W=15\tfrac{7}{8}\text{ lbs.}$$

The least weight, W' lbs. (suspended at A), is found by removing the movable weight, and then we have

$$W'\times 4=2\tfrac{1}{4}\times 14,\quad \text{whence}\quad W'=7\tfrac{7}{8}\text{ lbs.}$$

Or, if as in example 8, the movable weight can pass C, we have

$$(W'+1)\,4=2\tfrac{1}{4}\times 14,\quad \text{whence}\quad W'=6\tfrac{7}{8}\text{ lbs.}$$

Also, since the movable weight is 1 lb., the graduations to shew pounds are at distances equal to AC, *i.e.* 4 ins.

11. Take the figure, p. 218, with $W=4W$, $W'=W$, $P=W$, and G the middle point of AB. Let the fulcrum be at distance of x ins. from A. The movable weight must be at B. We have

$$4W\times x=W(20-x)+W(40-x),\quad \text{whence}\quad x=10\text{ ins.}$$

12. The zero graduation is when $W=0$, and, therefore, is at the centre of gravity of the instrument. Thus the greatest weight is weighed by placing the fulcrum at the 19th division, and moments about this point, if P be the weight of the instrument, and a be the length of each of the 20 equal parts, give

$$P\times 19a=57\times a,\quad \text{whence}\quad P=3\text{ oz.}$$

13. If AG be the steelyard, and X, X_1 be the positions of the fulcrum in the two cases, then moments round the fulcrum give

$$4AX=1\,(AG-AX),\quad i.e.\quad AX=\frac{1}{5}AG;$$

and
$$5AX_1=1\,(AG-AX_1),\quad i.e.\quad AX_1=\frac{1}{6}AG;$$

$$\therefore XX_1=1\text{ inch}=\left(\frac{1}{5}-\frac{1}{6}\right)AG=\frac{1}{30}AG,\quad \text{so that}\quad AG=30\text{ ins.}$$

14. Take the figure, p. 220. The first graduation is at the middle point of AG, X_1 say; the second graduation is at X_2 say, where

$$AX_2=\frac{1}{3}AG;$$

hence, if the fulcrum, X say, be the middle point of X_1X_2, we have

$$AX=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right)AG=\frac{5}{12}AG;$$

$$\therefore XG=\frac{7}{12}AG;$$

$$\therefore W\times\frac{5}{12}AG=P\times\frac{7}{12}AG,\quad \text{whence}\quad W=\frac{7}{5}P.$$

15. Take the figure of Art. 171. If the marks on the machine denote equal increments W, we have

$$CO\,.\,1=CG\,.\,W', \text{ and } CX_1\,.\,1=CG\,.\,W'+CA\,.\,W,$$

i.e.
$$W\,.\,CA=OX_1\,.\,1=1\,.\,3CO.$$

Since
$$\frac{1}{2}W'\,.\,CG=\frac{1}{2}\,.\,1\,.\,CO=\frac{1}{6}W\,.\,CA,$$

it follows that a loss of $\frac{W'}{2}$ at G may be replaced by increasing the quantity weighed by one-sixth, *i.e.* the quantity weighed is greater by one-sixth than indicated by the original markings of the machine.

If the original markings denoted lbs. the increase must be $\frac{1}{6}$ lb.

16. Take the figure, p. 218. Let P at X balance nP at A; then

$$P\times CX=W'\times CG+nP\times CA;$$

when W' becomes $\frac{9W'}{10}$, $n'P$ is required at A, so that we have

$$P\times CX=\frac{9W'}{10}\times CG+n'P\times CA;$$

$$\therefore\ n'P-nP=\frac{W'}{10}\times\frac{CG}{CA},$$

i.e. the increase
$$=\frac{W'}{10}\times\frac{CG}{CA}=\frac{W'}{10}\,.\,\frac{x}{y},$$

if x and y are the distances of the centre of gravity of the machine and the end from the fulcrum.

17. Taking the figure of Art. 171, we have

$$W\,.\,CA+W'\,.\,CG=P\,.\,CX,$$

when the machine is correct.

If he increase the movable weight, the right-hand side of this equation becomes increased. Hence the left-hand side, and therefore W, is increased. But W was the quantity properly belonging to the graduations. Hence if he increase P he cheats himself.

If he, similarly, decrease P he cheats his customers.

18. Take the figure, p. 218. To weigh 1 stone (*i.e.* 14 lbs.), we have 7 oz. *i.e.* $\frac{7}{16}$ lb. at B, so that

$$14\times CA+W'\times CG=\frac{7}{16}\times 12+\frac{8}{16}\times CO.$$

To weigh 2 stones, we have 14 oz. at B.

$$\therefore\ 28\times CA+W'\times CG=\frac{14}{16}\times 12+\frac{8}{16}\times CO.$$

Hence, by subtraction, $14CA=\frac{7}{16}\times 12$, and $CA=\frac{3}{8}$ in.

Again, when the weights at A are respectively nothing and one lb., we have

$$W'.CG=\frac{1}{2}.CO \text{ and } 1.CA+W'.CG=\frac{1}{2}CX_1.$$

Hence, by subtraction,

$$CA=\frac{1}{2}.OX_1, \textit{ i.e. } OX_1=2.CA=\frac{3}{4}\text{ inch}=X_1X_2 \text{ etc.}$$

19. Take the figure, p. 218. In weighing 1 stone, placed at A, we have the sliding weight, $\frac{n}{16}$ lb., at O (zero point), and $\frac{m}{16}$ lb. at B; then

$$14\times CA+W'\times CG=\frac{m}{16}\times 12\text{ (ins.)}+\frac{n}{16}\times CO\text{............(i).}$$

In weighing 1 stone 1 lb., we have the sliding weight at X_1, and $\frac{m}{16}$ lb. at B; then

$$15\times CA+W'\times CG=\frac{m}{16}\times 12+\frac{n}{16}\times CX_1\text{............(ii).}$$

In weighing 2 stones 1 lb., we have the sliding weight at X_1, and $\frac{2m}{16}$ lb. at B; thus

$$29\times CA+W'\times CG=\frac{2m}{16}\times 12+\frac{n}{16}\times CX_1\text{........... (iii).}$$

Hence, by subtraction, $CA=\frac{n}{16}\times OX_1$, from (i) and (ii),

and $$14CA=\frac{m}{16}\times 12, \text{ from (ii) and (iii).}$$

Therefore, by division, $\frac{1}{14}=\frac{n}{12m}\times OX_1$,

whence $$OX_1=\frac{6m}{7n}\text{ ins.}=X_1X_2=\&c.$$

EXAMPLES. XXX. (Pages 230, 231.)

1. Here $$\frac{25}{W}=1\tfrac{1}{2}\div(2\pi\times 3\tfrac{1}{2}\times 12),$$

whence $$W=4400\text{ lbs.}$$

2. Here $$216=\frac{W}{P}=2\pi l\div\frac{1}{6},$$

whence $l=$ the required length $=5\tfrac{5}{11}$ inches.

3. Here $$\frac{P}{1100}=\left(\frac{1}{7}\times\frac{2}{3}\right)\div(2\pi\times 18),$$

whence $$P=\frac{25}{27}\text{ lb. wt.}$$

4. Here $$\frac{P}{15\times 112}=\frac{1}{4}\div(2\pi\times 4\times 12),$$

whence $$P=1\tfrac{69}{176}\text{ lb. wt.}$$

5. Here $$\frac{P}{2240}=\frac{1}{2}\div(2\pi\times 36),$$

whence $$P=4\tfrac{94}{99}\text{ lbs. wt.}$$

6. Here $$\frac{W}{50}=(2\pi\times 24)\div\frac{1}{4},$$

whence $$W=13\tfrac{23}{49}\text{ tons wt.}$$

7. Here $$\frac{W}{112}=(2\pi\times 2)\div\frac{1}{10},$$

whence $$W=6\tfrac{2}{7}\text{ tons wt.}$$

8. Here $$\frac{P}{2240}=\left(\frac{1}{7}\times 1\right)\div(2\pi\times 1),$$

whence $$P=50\tfrac{10}{11}\text{ lbs. wt.}$$

9. Here $\dfrac{1000}{10}=\dfrac{2\pi\times 6}{\text{distance}}$, whence distance $=\dfrac{66}{175}$ ft. $=4\tfrac{92}{175}$ ins.

10. Mechanical advantage $=\dfrac{W}{P}=(2\pi\times 2\times 12)\div\left(\dfrac{1}{5}-\dfrac{1}{6}\right)=4525\tfrac{5}{7}$.

11. Mechanical advantage $=\dfrac{W}{P}=(2\pi\times 12)\div\left(\dfrac{1}{8}-\dfrac{1}{9}\right)=5430\tfrac{6}{7}$.

12. The door rises through $\dfrac{1}{4}$ of the distance between two consecutive threads, since a right angle is $\dfrac{1}{4}$ of a revolution. Hence the work done $=\left(\dfrac{1}{4}\times 100\times\dfrac{2}{12}\right)$ ft.-lbs. $=4\tfrac{1}{6}$ ft.-lbs.

Page 234.

Ex. 1. (1) Horizontally backwards, so as to cause the wheel to rotate.

(2) Forwards, so as to prevent the foot slipping backwards.

Ex. 2. If F be the force of friction, resolving horizontally, we have

$$F=10\cos 30^\circ=\frac{10\sqrt{3}}{2}=8\cdot 66 \text{ lbs. wt.}$$

Ex. 3. The resultant of forces of 7 lbs. wt. and 8 lbs. wt. acting at an angle of 60°

$$=\sqrt{7^2+8^2+2\,.\,7\,.\,8\cos 60^\circ}=\sqrt{169}=13 \text{ lbs. wt.}=F.$$

Also, if α be the required angle, we have

$$\frac{\sin\alpha}{8}=\frac{\sin 60^\circ}{13},$$

so that $$\sin\alpha=\frac{4\sqrt{3}}{13},\ i.e.\ \alpha=\sin^{-1}\frac{4\sqrt{3}}{13}.$$

Ex. 4. Resolving along the plane, we have

(1) $F+14=40\sin 30^\circ=20$; hence $F=6$ lbs. wt.

(2) $F+40\sin 30^\circ=25$; hence $F=25-20=5$ lbs. wt.

(3) $$F+20\cos 30^\circ=40\sin 30^\circ=20;$$

hence $$F=20-10\sqrt{3}=2\cdot 68 \text{ lbs. wt.}$$

(4) $$F+40\sin 30^\circ=30\cos 30^\circ;$$

hence $$F=15\sqrt{3}-20=5\cdot 98 \text{ lbs. wt.}$$

In all the cases the force necessary to support 40 lbs. on a smooth plane $=40\sin 30^\circ=20$ lbs. wt., whence we see at once in which direction to place F.

EXAMPLES. XXXI. (Pages 244 - 246.)

1. (1) Resolving horizontally and vertically, if P lbs. wt. be the required force, we have $P=\mu R$, and $R=40$;

$$\therefore\ P=\cdot 25\times 40=10 \text{ lbs. wt.}$$

(2) Again, if $\alpha=\cos^{-1}\frac{3}{5}$, $\cos\alpha=\frac{3}{5}$, and $\sin\alpha=\frac{4}{5}$;

resolving horizontally and vertically, we have

$$P\cos\alpha=\mu R, \text{ and } R+P\sin\alpha=40;$$

$$\therefore\ P\times\frac{3}{5}=\cdot 25\left(40-P\times\frac{4}{5}\right),$$

whence $$P=12\tfrac{1}{2} \text{ lbs. wt.}$$

Also the resultant pressure is the resultant of R and μR in each case; hence

(1) the resultant pressure $=\sqrt{(40)^2+(10)^2}=10\sqrt{17}$ lbs. wt., at an angle $\tan^{-1}\frac{R}{\mu R}$, *i.e.* $\tan^{-1}4$ with the horizontal;

(2) since

$$R=4P\cos\alpha=30 \text{ lbs. wt., and } \mu R=\frac{30}{4} \text{ lbs. wt.,}$$

the resultant pressure

$$=\frac{15}{2}\sqrt{4^2+1^2}=\frac{15}{2}\sqrt{17} \text{ lbs. wt.,}$$

in the same direction as (1).

2. If W be the weight of the block, and P be the force, then, resolving horizontally and vertically, we have

$$P\cos 45^\circ=\mu R, \text{ and } R+P\sin 45^\circ=W;$$

hence
$$\frac{P}{\sqrt{2}}=\cdot 5\left(W-\frac{P}{\sqrt{2}}\right); \text{ whence } 3P=W\sqrt{2},$$

i.e.
$$\frac{P}{W}=\frac{\sqrt{2}}{3}=\cdot 4714.$$

3. Resolving horizontally and vertically, we have

$$\mu R=10, \text{ and } R=30; \text{ hence } \mu=\frac{1}{3}.$$

The resultant reaction

$$=\sqrt{(30)^2+(10)^2}=10\sqrt{10} \text{ lbs. wt.,}$$

at an angle $\tan^{-1}\frac{R}{\mu R}$, *i.e.* $\tan^{-1}3$ with the horizon.

4. If the required force P act at an angle θ with the plane, then, resolving horizontally and vertically, we have

$$P\cos\theta=\mu R, \text{ and } R+P\sin\theta=W;$$

$$\therefore\ P(\cos\theta+\mu\sin\theta)=\mu W,$$

or
$$P\cos(\theta-\phi)=W\sin\phi, \text{ if } \mu=\tan\phi.$$

$$\therefore\ P=\frac{W\sin\phi}{\cos(\theta-\phi)};$$

this is least when
$$\cos(\theta-\phi)=1,$$

i.e. when $\theta=\phi$, and then $P=W\sin\phi$.

5. If
$$\alpha=\cos^{-1}\frac{12}{13}, \text{ then } \cos\alpha=\frac{12}{13},$$

and
$$\sin\alpha=\frac{5}{13}.$$

Also, if P lbs. wt. be the required force, then resolving parallel and perpendicular to the plane, we have

$$P+\mu R=112\sin\alpha, \text{ and } R=112\cos\alpha;$$

whence

$$P=112(\sin\alpha-\mu\cos\alpha)=112\left(\frac{5}{13}-\frac{4}{13}\right)=8\tfrac{8}{13} \text{ lbs. wt.}$$

In the second case μR acts downwards; hence, changing the sign of μ, we have

$$P=112\left(\frac{5}{13}+\frac{4}{13}\right)=77\tfrac{7}{13} \text{ lbs. wt.}$$

6. If a be the inclination of the plane to the horizon, $\tan a=\frac{3}{4}$, and therefore

$$\sin a=\frac{3}{5}, \quad \cos a=\frac{4}{5}.$$

Resolving parallel and perpendicular to the plane, we have

$$8+\mu R=20 \sin a, \quad \text{and} \quad R=20 \cos a;$$

$$\therefore 8+\mu \times 20\times\frac{4}{5}=20\times\frac{3}{5}, \quad \text{whence} \quad \mu=\frac{1}{4}.$$

7. Since the body rests in limiting equilibrium on a plane whose slope is 30°, we have

$$\mu=\tan 30^\circ=\frac{1}{\sqrt{3}}.$$

Also, if P lbs. wt. be the required force, then resolving parallel and perpendicular to the plane, we have

$$P+\mu R=4 \sin 60^\circ=2\sqrt{3}, \quad \text{and} \quad R=4\cos 60^\circ=2;$$

hence $$P=2\sqrt{3}-\frac{2}{\sqrt{3}}=\frac{4}{\sqrt{3}}=\frac{4}{3}\sqrt{3} \text{ lbs. wt.}$$

8. Let a be the inclination of the plane to the horizon; then

$$\tan a=\frac{3}{4}.$$

Since the weight just rests on a plane at an angle $\tan^{-1}\frac{3}{4}$, we have $\mu=\frac{3}{4}$; and if P lbs. wt. be the required force, then resolving parallel and perpendicular to the plane, we have

$$P-\mu R=30 \sin a, \quad \text{and} \quad R=30 \cos a;$$

hence

$$P=30 \sin a+\tan a\times 30\cos a=60 \sin a=60\times\frac{3}{5}=36 \text{ lbs. wt.}$$

9. If a be the inclination of the plane to the horizon, and R and R' be the normal reactions in the two cases, then resolving parallel and perpendicular to the plane, we have

(1) $$24+\mu R=60 \sin a, \quad \text{and} \quad R=60\cos a;$$

whence $$5\mu \cos a=5\sin a-2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

(2) $$36-\mu R'=60\sin\alpha, \text{ and } R'=60\cos\alpha;$$

hence $$R'=R, \text{ and } 60\sin\alpha-24=36-60\sin\alpha.$$

$$\therefore \sin\alpha=\frac{1}{2}, \text{ i.e. } \alpha=30^\circ,$$

and, from (i), $$\mu=\tan\alpha-\frac{2}{5}\sec\alpha=\frac{1}{\sqrt{3}}-\frac{2}{5}\cdot\frac{2}{\sqrt{3}}=\frac{1}{5\sqrt{3}}=\frac{\sqrt{3}}{15}.$$

10. If α and β be the inclinations to the horizon of the rough and smooth planes respectively, W be each weight, and R be the normal reaction of the rough plane, then resolving parallel to the plane with inclination β, T being the tension of the string, we have $T=W\sin\beta$; also, resolving parallel and perpendicular to the plane with inclination α, we have

$$T-\mu R=W\sin\alpha, \text{ and } R=W\cos\alpha;$$

hence, eliminating T and R, we have

$$W\sin\beta=W\sin\alpha+\mu W\cos\alpha, \text{ i.e. } \sin\beta=\sin\alpha+\mu\cos\alpha.$$

11. Let W_1 and W_2 be the weights ($W_1>W_2$), R and R' be the normal reactions, T be the tension of the string, and α be the inclination of the plane to the horizon. Then, resolving parallel to the plane, we have

$$T+\mu R=W_1\sin\alpha, \text{ and } T-\mu R'=W_2\sin\alpha;$$

also, resolving perpendicular to the plane, we have

$$R=W_1\cos\alpha, \text{ and } R'=W_2\cos\alpha;$$

hence, eliminating T, R and R', we have

$$W_1\sin\alpha-W_2\sin\alpha=\mu W_1\cos\alpha+\mu W_2\cos\alpha,$$

whence $$\tan\alpha=\frac{\mu W_1+\mu W_2}{W_1-W_2}, \text{ i.e. } \alpha=\tan^{-1}\left(\mu\frac{W_1+W_2}{W_1-W_2}\right).$$

12. Let R and R' be the normal reactions of the planes with inclinations of 60° and 30°, respectively, and T be the tension of the string. The weight W on the more elevated plane is on the point of moving downwards. Resolving parallel and perpendicular to the plane with inclination of 60°, we have

$$T+\mu R=W\sin 60^\circ, \text{ and } R=W\cos 60^\circ;$$

and resolving parallel and perpendicular to the plane with inclination of 30°, we have

$$T-\mu R'=W\sin 30^\circ, \text{ and } R'=W\cos 30^\circ;$$

hence, eliminating T, R and R', we have

$$W\sin 60^\circ-\mu W\cos 60^\circ=W\sin 30^\circ+\mu W\cos 30^\circ,$$

i.e. $$\sqrt{3}-1=1+\mu\sqrt{3}, \text{ whence } \mu=2-\sqrt{3}.$$

13. The particle will be on the point of motion when the resultant reaction of the sphere just balances its weight; therefore when the normal at the point of contact makes the angle $\lambda = \tan^{-1}\mu$ with the vertical, *i.e.* when the radius from the position of the particle to the centre of the sphere makes an angle $\tan^{-1}\mu$ with the vertical.

√14. As in the last example, the radius from the position of the particle to the centre of the sphere must make an angle λ with the vertical in the extreme case of equilibrium, and here

$$\lambda = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ;$$

hence the height of the particle above the lowest point

$$= a\,(1 - \cos 30^\circ) = a\,(1 - \cdot 866\ldots) = a \times \cdot 134.$$

15. If P be the force of traction exerted by the horse; θ be the angle the traces make with the ground, supposed at an angle a to the horizon; R the normal reaction at any instant; μ $(=\tan\lambda)$ the coefficient of friction; and if W be the weight of the sledge; then, resolving parallel and perpendicular to the ground, we have

$$P\cos\theta = W\sin a + \mu R, \quad \text{and} \quad P\sin\theta = W\cos a - R;$$

$$\therefore\ P\,(\cos\theta + \mu\sin\theta) = W\,(\sin a + \mu\cos a),$$

i.e.

$$P\cos(\theta - \lambda) = W\sin(a + \lambda),$$

or

$$P = W\frac{\sin(a+\lambda)}{\cos(\theta-\lambda)};$$

hence P is least when $\cos(\theta - \lambda) = 1$, *i.e.* when $\theta = \lambda$.

16. Let A be the body, AB the direction of the force at 40° to the horizontal.

Let AC be the normal; on the side of AC away from AB draw AD so that $\angle CAD = 25^\circ$. Then AD is the direction of the resulting reaction when motion is just about to ensue.

Draw AK vertically downwards and equal to 5 ins.

[Scale 1 cwt. = 1 inch.]

Draw KE parallel to AB to meet DA produced in E; the AE represents the least value of P on the assumed scale.

17. Let A be the body on the given inclined plane AB; measure AC vertically downwards and equal to 8 inches to represent the weight [scale 14 lbs. = 1 inch]; up the plane measure $AB = 1\cdot 07$ inch to represent the force.

Complete the parallelogram $ACDB$; join DA and produce to E; AE will be found to be on the upper side of the normal AR to the plane and RAE will be the angle of friction. Measure off AR = one inch and erect a perpendicular RE to AR to meet AE in E. Then $\tan\epsilon = \dfrac{RE}{AR}$ is the coefficient of friction.

On the lower side of AR take AE' such that $\angle RAE' = \angle RAE$. Then AE' is the resultant direction of the reaction when the body is on the point of moving up the plane.

Produce $E'A$ to meet CD in D', and complete the parallelogram $ACD'B'$; then AB' represents the required minimum dragging force.

Page 249.

Ex. 2. Here $P=0$, so that $\alpha=\lambda$, *i.e.* $\tan\alpha = \cdot15 = \frac{3}{20}$.

Ex. 3. Here $2\pi a=\frac{3}{2}$, and $2\pi a\tan\alpha=\frac{1}{3}$. $\therefore\ \tan\alpha = \frac{1}{3}\div\frac{3}{2}=\frac{2}{9}$.

Also $$\tan\lambda = \cdot15 = \frac{3}{20}, \text{ and } b=12.$$

Hence the force which will just support the screw

$$=W\times\frac{3}{4\pi\times12}\tan(\alpha-\lambda)=\frac{W}{16\pi}\times\frac{\frac{2}{9}-\frac{3}{20}}{1+\frac{2}{9}\cdot\frac{3}{20}}=\frac{W}{16\pi}\times\frac{13}{186}=\frac{W}{16\pi}(\cdot07).$$

Also the force which will just move the screw

$$=\frac{W}{16\pi}\times\frac{\frac{2}{9}+\frac{3}{20}}{1-\frac{2}{9}\cdot\frac{3}{20}}=\frac{W}{16\pi}\times\frac{67}{174}=\frac{W}{16\pi}(\cdot385).$$

If the screw be smooth, $\lambda=0$, and the value of the power

$$=\frac{W}{16\pi}\times\frac{2}{9}=\frac{W}{16\pi}(\cdot\dot{2}).$$

Also the efficiency

$$=\frac{\text{force required when there is no friction}}{\text{actual force required}}=\frac{\cdot\dot{2}}{\cdot385}=\frac{2}{3\cdot465}=\cdot577.$$

EXAMPLE. (Page 254.)

If P be the force exerted on each end of the power arm, the work done in one revolution of the power arm is $2\times P\times2\pi b$, and the work done against the weight $=W.c$. These two are equal.

$$\therefore\ W:P::4\pi b:c \quad\text{..........................(1).}$$

In the case of the wedge, if P be the force exerted at the centre of the base of the wedge, and R the corresponding pressure on the face of the wedge, we have, by resolving along the direction of P,

$$P=R\sin\frac{\alpha}{2}=R\cdot\frac{c}{4\pi b},\ \textit{i.e. } R:P::4\pi b:c \quad\text{...........(2).}$$

The multiplications in the cases (1) and (2) are thus the same.

EXAMPLES. XXXII. (Pages 257—259.)

1. By Art. 197, the work done

$$=6\times 112\left(3+\frac{2}{15}\times 20\right)\text{ ft.-lbs.}=3808\text{ ft.-lbs.}$$

2. The height of the plane

$$=330\sin 30^\circ=165\text{ feet,}$$

and the base of the plane

$$=330\cos 30^\circ=165\sqrt{3}\text{ feet;}$$

$$\therefore\ \text{the work done}=10\times 2240\left(165+\frac{1}{\sqrt{3}}\times 165\sqrt{3}\right)\text{ ft.-lbs.}$$

$$=(22400\times 330)\text{ ft.-lbs.}=7392000\text{ ft.-lbs.}$$

Also, the time occupied being 30 minutes, the required H.P.

$$=\frac{22400\times 330}{33000\times 30}=\frac{224}{30}=7\tfrac{7}{15}.$$

3. Since all the water has to be raised to the top of the tank, the work done in filling it

$$=\left(24\times 12\times 16\times\frac{1000}{16}\times 80\right)\text{ ft.-lbs.}$$

$$=(288\times 80000)\text{ ft.-lbs.}=23040000\text{ ft.-lbs.}$$

Also, the required H.P.

$$=\frac{288\times 80000}{\cdot 5\times 33000\times 240}=5\tfrac{9}{11}.$$

4. If it raise x cubic feet, the work done

$$=\left(x\times\frac{1000}{16}\times 300\right)\text{ ft.-lbs.;}$$

this $=\pi\times(30)^2\times 8\times 15\times\cdot 65\times 11\times 60,$

whence $x=7766\cdot 12.$

5. The area of the piston is $\pi . 40^2$ and hence the thrust on it is $\pi . 40^2 . 12$ lbs. wt. If x be the efficiency of the engine, then

$$x\times(\pi . 40^2 . 12)\times(20 . 11)$$

$$=\text{work actually performed by the engine}$$

$$=42\tfrac{1}{2}\times 62\tfrac{1}{2}\times 3000,$$

since a cubic foot of water weighs $62\frac{1}{2}$ lbs.

Hence x.

6. If there were no friction, the force required would

$$=\frac{200\times\frac{1}{2}}{4}\text{ lbs.}=25\text{ lbs.}$$

Hence (Art. 198) the efficiency

$$=\frac{25}{56}=\cdot 446\ldots.$$

7. Had there been no friction, the efforts would have been $\frac{10}{4}$, $\frac{80}{4}$ and $\frac{160}{4}$, *i.e.* $2\frac{1}{2}$, 20 and 40 lbs. wt.

The corresponding efficiencies are thus

$$\frac{2\frac{1}{2}}{23},\ \frac{20}{58}\text{ and }\frac{40}{85},$$

i.e. ·11, ·34, and ·47 nearly.

8. We have the equations

$$12=a+700b,\text{ and }7\cdot 5=a+800b.$$

By solving, we get

$$a=4\cdot 125\text{ and }b=\cdot 01125.$$

9. On plotting the results the straight line joining the second and last points lies evenly amongst the others. Hence, taking these points, we have

$$\left.\begin{aligned}22\cdot 7&=a+b\times 180,\\ \text{and}\qquad 31\cdot 4&=a+b\times 270.\end{aligned}\right\}$$

On solving, $a=5\cdot 3$ and $b=\cdot 097$, nearly,

so that $P=5\cdot 3+\cdot 097W.$

A slightly different result would be obtained according to the points chosen to determine the representative straight line of the graph.

10. On plotting the results the straight line joining the first and last point but one lies evenly. Taking these two results and assuming the relation

$$P=a+bW,$$

we have $25=a+75b$ and $214=a+875b.$

Solving we have, approximately,

$$b=\cdot 236\text{ and }a=7\cdot 3.$$

$$\therefore\ P=7\cdot 3+\cdot 236W.$$

Also, since there are five strings, the value of $P_0=\frac{W}{5}$.

Also
$$E=\frac{P_0}{P}=\frac{W}{5P}=\frac{W}{36\cdot5+1\cdot18W},$$

and
$$M=\frac{W}{P}=\frac{W}{7\cdot3+\cdot236W}.$$

The graph of P_0 is a straight line; that of P is approximately a straight line and those of E and M approximately curves called hyperbolas.

11. Here the straight line joining the first and last points lies evenly. Taking these two results and assuming the law $P=a+bW$, we have $9=a+b$ and $56=a+11b$.

Hence, solving, we have $a=4\cdot3$ and $b=4\cdot7$, so that $P=4\cdot3+4\cdot7W$, where W is measured in tons and P in lbs

Also since the velocity ratio is 500, one ton is supported by $\frac{2240}{500}$, *i e.* 4 48 lbs.

Hence, P_0 being expressed in lbs. and W in tons, we have

$$P_0=4\cdot48W.$$

$$\therefore\ \text{Efficiency}=\frac{P_0}{P}=\frac{4\cdot48W}{4\cdot3+4\cdot7W}.$$

When $W=5$, the efficiency$=\frac{22\cdot4}{4\cdot3+23\cdot5}$

$$=\frac{224}{278}=\cdot806 \text{ nearly.}$$

When $W=10$, it$=\frac{44\cdot8}{4\cdot3+47}=\frac{44\cdot8}{51\cdot3}$

$$=\cdot878 \text{ nearly.}$$

12. On plotting the line joining the first and fourth lies evenly. Taking these two results and putting $P=a+bW$, we have

$$24=a+b \text{ and } 57=a+7b.$$

$$\therefore\ b=5\cdot5 \text{ and } a=18\cdot5,$$

so that
$$P=18\ 5+5\cdot6W,$$

where P is measured in lbs. and W in tons.

Also the velocity ratio$=\frac{2\pi\,.\,15}{\frac{1}{4}}=120\pi$

$$=377 \text{ nearly.}$$

Therefore 1 ton would be supported by a force of $\frac{2240}{377}$ lbs., *i.e.* 5·94 lbs., if there were no friction.

$\therefore$ $P_0 = 5{\cdot}94W$, P_0 being also measured in lbs.

$$\therefore \text{ Efficiency} = \frac{P_0}{P} = \frac{5{\cdot}94W}{18{\cdot}5 + 5{\cdot}6W}.$$

When $W = 4$, the efficiency

$$= \frac{23{\cdot}76}{18{\cdot}5 + 22{\cdot}4} = \frac{23{\cdot}76}{40{\cdot}9} = {\cdot}581 \text{ nearly.}$$

When $W = 9$, it

$$= \frac{53{\cdot}46}{18{\cdot}5 + 50{\cdot}4} = \frac{53{\cdot}46}{68{\cdot}9} = {\cdot}776 \text{ nearly.}$$

EXAMPLES. XXXIII. (Pages 262, 263.)

1. Take the figure ~~p. 260~~, but the wall being smooth there is no friction $\mu' S$ at B.

Now $$OB^2 = AB^2 - AO^2 = (13)^2 - 5^2,$$

so that $$OB = 12 \text{ feet.}$$

Resolving horizontally, we have

$$\mu R = S;$$

taking moments about A, we have

$$56 \times 2\tfrac{1}{2} = S \times 12, \text{ whence } S = \frac{35}{3},$$

so that $$\mu R = \frac{35}{3} = 11\tfrac{2}{3} \text{ lbs. wt.}$$

2. As in Art. 207,

$$\tan\theta = \frac{1 - \mu\mu'}{2\mu} = \left[1 - \frac{3}{7} \cdot \frac{1}{3}\right] \div \frac{6}{7} = 1;$$

i.e. $$\theta = 45^\circ.$$

3. Here, with $\mu = \mu' = \frac{1}{3}$, we have

$$\tan\theta = \left[1 - \frac{1}{3} \cdot \frac{1}{3}\right] \div \frac{2}{3} = \frac{4}{3}, \text{ i.e. } \theta = \tan^{-1}\frac{4}{3};$$

hence the inclination to the vertical

$$= \frac{\pi}{2} - \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{3}{4}.$$

4. Here, with $\mu' = 0$, the wall being smooth, we have

$$\tan\theta = \frac{1}{2\mu}, \text{ i.e. } \cot\theta = 2\mu;$$

hence the inclination to the vertical $= \tan^{-1}(2\mu)$.

Or thus: Let AB be the ladder, and let the normal reaction at B and the direction of W the weight of the ladder meet in D, then the reaction at A must also pass through D; also if AC be parallel to OB meeting the horizontal line through B in C, then CAD = the angle of friction for the floor, *i.e.* $\tan CAD = \mu$.

Now $$\tan BAC = \frac{BC}{AD} = \frac{2CD}{AD} = 2\tan CAD = 2\mu,$$

so that $$\angle BAC = \tan^{-1}(2\mu).$$

5. Take the figure 1 ~~p. 260~~, but with ϕ the inclination of the ladder to the ground; also here $\mu = \mu'$.

Resolving horizontally and vertically, we have

$$\mu R = S, \text{ and } R + \mu S = W.$$

Taking moments about A, we have

$$W \cdot a\cos\phi = \mu S \cdot 2a\cos\phi + S \cdot 2a\sin\phi,$$

or $$(R + \mu^2 R)\cos\phi = \mu^2 R \cdot 2\cos\phi + \mu R \cdot 2\sin\phi,$$

$$\therefore\ (1 - \mu^2)\cos\phi = 2\mu\sin\phi,$$

$$\therefore\ \cot\phi = \frac{2\mu}{1-\mu^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta.$$

$$\therefore\ \phi = \frac{\pi}{2} - 2\theta;$$

hence the inclination to the vertical $= 2\theta$.

If a man of weight W' can just ascend a distance x before the ladder slips, we have

$$\mu R = S, \text{ and } R + \mu S = W + W',$$

whence

$$S = (W + W')\sin\theta\cos\theta, \text{ and } \mu S = (W + W')\sin^2\theta.$$

Taking moments about A, we have

$$W' \cdot x\cos\phi + W \cdot a\cos\phi = S \cdot 2a\sin\phi + \mu S \cdot 2a\cos\phi.$$

$$\therefore\ W' \cdot x\sin 2\theta + W \cdot a\sin 2\theta = (W + W')a\sin 2\theta\cos 2\theta + (W + W')\,2a\sin^2\theta\sin 2\theta.$$

$$\therefore\ W'x = (W + W')a\cos 2\theta - Wa + (W + W')a(1 - \cos 2\theta) = W'a, \text{ i.e. } x = a;$$

hence the ladder can be ascended as far as the centre.

6. Take the figure 1 ~~p. 260~~, but with the wall smooth. Let the man of weight W' ascend a distance x. Resolving horizontally and vertically, we have

$$\mu R = S, \text{ and } R = W + W'.$$

Taking moments about A, we have

$$W \,.\, a\cos\theta + W' \,.\, x\cos\theta = S \,.\, 2a\sin\theta,$$

$$\therefore\ (Wa + W'x)\cos\theta = \mu\,(W + W')\,2a\sin\theta,$$

i.e. $$Wa + W'x = \mu\,(W + W')\,2a\tan\theta;$$

(but by example 4, p. 222,)

$$\left(\theta = \frac{\pi}{2} - \tan^{-1}(2\mu),\ \text{so that } \tan\theta = \frac{1}{2\mu};\right)$$

$$\therefore\ Wa + W'x = Wa + W'a,$$

i.e. $x = a$, in extreme position.

7. Take the figure p. 260, with the wall smooth and $\theta = 45°$. Let the man of weight W ascend a distance x. Resolving horizontally and vertically, we have

$$\mu R = S, \text{ and } R = W + W.$$

Taking moments about A, we have

$$W \,.\, a\cos 45° + W \,.\, x\cos 45° = S \,.\, 2a\sin 45°.$$

$$\therefore\ Wa + Wx = 2Sa;$$

but $$R = 2W, \text{ and } S = \mu R = \frac{3}{4}\times 2W = \frac{3W}{2}.$$

$$\therefore\ a + x = 3a,\ i.e.\ x = 2a.$$

8. Take the figure p. 260, with $\theta = 45°$, $\mu = \frac{1}{2}$ and $\mu' = \frac{1}{3}$. Let the man of weight $\frac{W}{2}$ ascend a distance x. Resolving horizontally and vertically, we have

$$\frac{R}{2} = S. \text{ ....(i), and } R + \frac{S}{3} = W + \frac{W}{2} = \frac{3W}{2} \text{ ....(ii).}$$

Taking moments about A, we have

$$W \,.\, a\cos 45° + \frac{W}{2} \,.\, x\cos 45° = S \,.\, 2a\sin 45° + \frac{S}{3} \,.\, 2a\cos 45°,$$

or $$\frac{W}{2}(2a + x) = \frac{4S}{3} \,.\, 2a = \frac{2R}{3} \,.\, 2a, \text{ by (i);}$$

but, from (i) and (ii),

$$\frac{3W}{2} = R + \frac{R}{6} = \frac{7R}{6};$$

$$\therefore\ \frac{7R}{18}(2a + x) = \frac{2R}{3} \,.\, 2a,\ i.e.\ 7\,(2a + x) = 12 \,.\, 2a,$$

whence $$x = \frac{5}{7} \,.\, 2a = \frac{5}{7}\times 70 = 50 \text{ feet.}$$

9. Let R be the normal reaction at the foot of either ladder, S the (symmetrically) horizontal reaction between the ladders at the top, and W be the required weight. W will produce a vertical pressure downwards on each ladder equal to $\frac{W}{2}$; hence, for either ladder, resolving horizontally and vertically, we have

$$\mu R = S, \text{ and } R = w + \frac{W}{2}.$$

Taking moments about the foot of either ladder, we have

$$w \cdot a \sin \alpha + \frac{W}{2} \cdot 2a \sin \alpha = S \cdot 2a \cos \alpha,$$

where $2a$ is the length of a ladder.

Hence $$\frac{w+W}{2} \tan \alpha = S = \mu R = \mu \cdot \frac{2w+W}{2};$$

whence $$W = w \cdot \frac{2\mu - \tan \alpha}{\tan \alpha - \mu}.$$

10. The frictions act opposite to the directions in which the ends of the ladder would begin to move. Take a figure as on p. 221, with μR and $\mu' S$ acting in reversed directions, and let P be the required force acting at A towards O. Then resolving horizontally and vertically, we have

$$S + \mu R = P \quad \dots\dots\dots\dots\dots\dots (1),$$

and $$W + \mu' S = R \quad \dots\dots\dots\dots\dots\dots (2).$$

Also, taking moments about A, we have

$$W \cdot a \cos 45^\circ + \mu' S \cdot 2a \cos 45^\circ = S \cdot 2a \sin 45^\circ$$

$$\therefore \; W + 2\mu' S = 2S \quad \dots\dots\dots\dots (3).$$

From (1) and (2), we have

$$P = S + \mu W + \mu\mu' S;$$

and from (3), $$S = \frac{W}{2(1-\mu')};$$

$$\therefore \; P = \mu W + (1 + \mu\mu') \frac{W}{2(1-\mu')}$$

$$= \frac{W}{2} \cdot \frac{2\mu - 2\mu\mu' + 1 + \mu\mu'}{1-\mu'} = \frac{W}{2} \cdot \frac{1 + 2\mu - \mu\mu'}{1-\mu'}$$

11. Let AB, BC be the ladders, B being the common vertex. When they are inclined at an angle α to the vertical, with an extra weight W at the middle point of AB, let F and G be the horizontal forces of friction at A and C, and let R, S be the vertical reactions at these points. Taking moments about A and C, we easily have

$$R=\frac{4w+3W}{4} \text{ and } S=\frac{4w+W}{4}.$$

Also, taking moments about B for the rod BA, we have

$$F.2a\cos\alpha+(w+W)a\sin\alpha=R.2a\sin\alpha,$$

so that

$$F=\frac{2w+W}{4}\tan\alpha.$$

So, taking moments about B for BC, we have

$$G.2a\cos\alpha+w.a\sin\alpha=S.2a\sin\alpha,$$

so that

$$G=\frac{2w+W}{4}\tan\alpha.$$

Hence $$\frac{F}{R}=\tan\alpha\,\frac{2w+W}{4w+3W} \text{ and } \frac{G}{S}=\tan\alpha\,\frac{2w+W}{4w+W}.$$

Therefore clearly $\frac{G}{S}>\frac{F}{R}$, so that the ratio of the friction to the normal reaction is greater at C than at A. Hence the equilibrium will become limiting at C before A. Hence C will slip first.

EXAMPLES. XXXIV. (Pages 264, 265.)

1. With the figure and notation of Art. 208, we have

$$\theta = \phi, \quad i.e. \ \tan\phi = \frac{1}{2}, \quad i.e. \ \phi = \tan^{-1}\frac{1}{2}.$$

Also $$\frac{2r}{h} = \frac{1}{2}\,(=\mu), \quad \text{so that } h = 4r,$$

i.e. the height = twice the diameter of the base.

2. Take the figure of Art. 208, with $\phi = 0$. Let W be the weight of the cylinder, and P be the horizontal force acting at F, the centre of ED, in the direction FD. Resolving vertically and horizontally, we have $R = W$, and $P = \mu R\ (=\mu W)$, R being the action at B.

There are two opposite couples, one of moment $P.FC$, *i.e.* μWh, tending to tip the cylinder over, and the other of moment $R.BC$, *i.e.* Wr, tending to keep it upright. Hence the cylinder will remain upright, *i.e.* will slide, if

$$Wr > \mu Wh, \quad i.e. \text{ if } r > \mu h, \quad i.e. \text{ if } \mu < \frac{r}{h}.$$

3. Let BC be the base of the equilateral triangle ABC, and P be the horizontal force at the vertex A. For turning about C, take moments about C, and we have

$$P.a\sin 60^\circ = W.a\cos 60^\circ,$$

where W is the weight and a the side of the triangle;

$$\therefore P = \frac{W}{\sqrt{3}} = \frac{1}{3}W\sqrt{3}.$$

For sliding, resolving vertically and horizontally, we have

$$P = \mu R, \quad \text{and} \quad R = W,$$

where R is the reaction at C;

$$\therefore P = \mu W.$$

The triangle will slide before turning if the second value of P be less than the first, *i.e.* if $\mu < \frac{1}{3}\sqrt{3}$.

4. Let C be the top of the sugarloaf, A be its lowest point, D be the centre of its base, and G be its centre of gravity. Then

$$GD = \frac{1}{4}CD; \quad \text{but} \quad CD = 4AD;$$

therefore $GD = AD$. When the sugarloaf is on the point of falling over, GA is vertical, and if θ be the required inclination, we have

$$\tan\theta = \tan AGD = \frac{AD}{GD} = 1, \quad i.e. \ \theta = 45^\circ.$$

5. If C be the vertex of the cone, A be its lowest point, D be the centre of its base, and G be its centre of gravity, then $GD=\frac{1}{4}CD$. The cone will be on the point of sliding when θ the inclination of the plane to the horizon is equal to the angle of friction. Also, when the cone is on the point of toppling over, GA is vertical.

Hence $$\tan\theta=\tan AGD=\frac{AD}{GD}=\frac{AD}{\frac{1}{4}CD}=4\tan\alpha,$$

and the initial motion of the cone will be sliding or toppling according as μ, the coefficient of friction, is $<$ or $>$ $4\tan\alpha$.

6. Let C be the vertex of the cone, A be its lowest point, D be the centre of its base, and G be its centre of gravity. If the cone be on the point of slipping, the inclination of the plane to the horizon

$$=\text{the angle of friction}=\tan^{-1}\frac{1}{\sqrt{3}}=30^\circ.$$

If the cone be at the same time on the point of turning over, GA is vertical.

Let 2α be the vertical angle of the cone; then we have

$$\tan\alpha=\frac{AD}{CD}=\frac{AD}{4DG};$$

but $$\frac{AD}{DG}=\cot DAG=\cot 60^\circ=\frac{1}{\sqrt{3}}.$$

$$\therefore \tan\alpha=\frac{1}{4\sqrt{3}},\quad i.e.\quad 2\alpha=2\tan^{-1}\frac{1}{4\sqrt{3}}=2\tan^{-1}\frac{\sqrt{3}}{12}.$$

7. The tension of the cord acts horizontally, and if A be the vertex of the cone, a be the radius of its base, B be the point on the latter nearest the pulley, and W be the weight of the cone, then (i) if the cone be on the point of turning over, the reaction of the table acts through B, and moments about B, if T_1 be then the tension of the cord, give $W\times CB=T_1\times AC$, where C is the middle point of the base;

i.e. $$W\times a=T_1'\times a\cot\alpha,\ \text{ or }\ T_1=W\tan\alpha.$$

(ii) If the cone be on the point of slipping, and if R be the vertical resistance of the table, μ be the coefficient of friction, and T_2 be then the tension of the cord we have, by resolving vertically and horizontally,

$$R=W,\ \text{ and }\ \mu R=T_2;\ \ i.e.\ \ T_2=\mu W;$$

hence the cone will turn over or slide first, according as $T_1 <$ or $> T_2$, *i.e.* according as $W\tan\alpha$ is $<$ or $>$ μW, *i.e.* according as μ is $>$ or $<$ $\tan\alpha$.

8. If G be the centre of gravity of the block, and A be the lowest point of its principal vertical section perpendicular to the plane, then the block will be on the point of toppling over when GA is vertical, and then the inclination of the plane to the horizon is 45°. Hence the limiting angle of friction must be not less than 45°, or the block will slip first, *i.e.* the least value of the coefficient of friction

$$=\tan 45° = 1.$$

9. If when the plane has been tilted through an angle θ the lamina be on the point of sliding down, then $\tan\theta = \mu$. If, however, when the plane has been tilted through an angle ϕ, the lamina be on the point of toppling over, then G, its centre of gravity, must be vertically above B, and the line BD, which joins B and the middle point D of AC, must be vertical. Also, since $BD = DC = DA$, the $\angle DBA =$ the $\angle BAC$. Hence ϕ, the inclination of the plane to the horizon, must $= \angle A$; and the lamina will begin to slide or topple over according as $\theta <$ or $> \phi$, *i.e.* as $\mu <$ or $> \tan A$.

10. Since the plane is perfectly rough, the plate will not slide, but will turn about B when the moment about B of the tension of the string (which is equal to w) is greater than that of W; *i.e.* when $w \cdot AB \cos\alpha > \frac{W}{2}$ (the sum of the horizontal distances of B from A and C), *i.e.* $> \frac{W}{2}(AB\sin\alpha + BC\cos\alpha)$, *i.e.* since $BC = AB$, when

$$w\cos\alpha > \frac{W}{2}(\sin\alpha + \cos\alpha), \quad \text{or} \quad w > \frac{W}{2}(1 + \tan\alpha).$$

11. If G be the centre of gravity of the block when on the point of toppling, it being supposed that the board is rough enough to prevent slipping and that its weight may be neglected, and if A represent in section the extremity of the board, B the lowest point of the block and C the centre of its base, and if α be the angle that BC makes with the horizon, then

$$\tan\alpha = \frac{BC}{GC} = \frac{3}{4},$$

and therefore $$\sin\alpha = \frac{3}{5};$$

the distance through which G has been raised

$$= BG + AB\sin\alpha - GC$$

$$= 3(\operatorname{cosec}\alpha + \sin\alpha) - 4 = 3\left(\frac{5}{3} + \frac{3}{5}\right) - 4 = \frac{14}{5};$$

hence the work done

$$= 2240 \times \frac{14}{5} = 6272 \text{ ft.-lbs.}$$

Page 267.

Ex. 1. Take the figure p. 267, with

$$W_1 = W_2 = W, \quad \mu_1 = \frac{1}{3}, \text{ and } \mu_2 = \frac{1}{2}.$$

If θ be the required inclination of the plane when both weights W are on the point of moving dównwards, R_1 and R_2 the normal reactions of the bodies, (the smoother body being the lower, or else the string might be slack), and if T be the tension of the string, then for the equilibrium of each weight separately, we have, resolving along and perpendicular to the plane,

$$W \sin\theta = T + \frac{R_1}{3}, \text{ and } W\cos\theta = R_1;$$

$$\therefore T = W\left(\sin\theta - \frac{\cos\theta}{3}\right).$$

Also $$W\sin\theta+T=\frac{R_2}{2},\ \text{ and }\ W\cos\theta=R_2;$$

$$\therefore\ T=W\left(\frac{\cos\theta}{2}-\sin\theta\right).$$

Hence $$\sin\theta-\frac{\cos\theta}{8}=\frac{\cos\theta}{2}-\sin\theta,\ \text{ or }\ 2\sin\theta=\frac{5}{6}\cos\theta,$$

$$\therefore\ \tan\theta=\frac{5}{12},\ \ \textit{i.e.}\ \ \theta=\tan^{-1}\frac{5}{12}.$$

Ex. 2. If θ be the greatest angle, and W be the weight of each body, the normal reactions upon the bodies each $=W\cos\theta$, as in Ex. 1; hence the sum of the frictional forces

$$=W\cos\theta\left(\frac{1}{2}+\frac{5}{8}+\frac{3}{8}\right)=\frac{3W}{2}\cos\theta,$$

and this has to support the components of the three weights down the plane;

hence we have $$\frac{3W}{2}\cos\theta=3W\sin\theta.$$

$$\therefore\ \tan\theta=\frac{1}{2},\ \ \textit{i.e.}\ \ \theta=\tan^{-1}\frac{1}{2}.$$

EXAMPLES. XXXV. (Pages 269—273.)

1. Take the figure p. 260, with $AG=a$, and $BG=b$. Resolving horizontally and vertically, we have

$$\mu R=S,\ \text{ and }\ R+\mu' S=W.$$

Also, taking moments about A, we have

$$W.\,a\cos\theta=\mu' S(a+b)\cos\theta+S(a+b)\sin\theta,$$

$$\therefore\ Wa=\mu' S(a+b)+\mu R(a+b)\tan\theta.$$

$$\therefore\ (R+\mu\mu' R)\,a=\mu\mu' R(a+b)+\mu R(a+b)\tan\theta.$$

$$\therefore\ \mu(a+b)\tan\theta=a+a\mu\mu'-a\mu\mu'-b\mu\mu',$$

whence $$\tan\theta=\frac{a-b\mu\mu'}{\mu(a+b)}.$$

2. Let AB be the rod of length l, and AC and BC be the planes, inclined at angles $\frac{\pi}{2}-\alpha$ and α respectively to the horizon. Normals AE and BE intersect on the circle on AB as diameter. If B be on the point of moving down, the reactions at A and B act along AO and BO, where the $\angle EAO=$ the $\angle EBO=\lambda$; similarly, if A be on the point of moving down, the reactions act along AO' and BO', where the $\angle EAO'=$ the $\angle EBO'=\lambda$. The two points O and O'

clearly lie on the above circle. Then the limiting positions of the weight on AB are H and K, where $O'H$ and OK are perpendicular to AB. Now the chord OO' subtends an angle 2λ at the circumference of the circle, so that $OO'=l\sin 2\lambda$; also OO' is parallel to the tangent at E, and therefore inclined at an angle $\frac{\pi}{2}-2\alpha$ to the horizon; hence

$$HK=l\sin 2\lambda\cos\left(\frac{\pi}{2}-2\alpha\right)=l\sin 2\lambda\sin 2\alpha.$$

α can be used for the inclination of either plane, since the substitution of $\frac{\pi}{2}-\alpha$ for α does not affect the result.

3. Let A and C be the two pegs, A being the lower one. If the rod be under A and over C, the centre of gravity G of the rod must be above AC; for otherwise there would be no pressure at A and the rod would swing about C.

The rod is clearly least when one end is just under A and it is on the point of slipping.

The reactions at A and C being R and S, and μ being the coefficient of friction, we have, by resolving along and perpendicular to the rod,

$$\mu R+\mu S=W\sin\alpha \qquad\qquad (1),$$

and

$$S=R+W\cos\alpha \qquad\qquad (2).$$

Also, by taking moments about A, we have

$$S\,.\,a=W\,.\,AG\cos\alpha \qquad\qquad (3).$$

From (1) and (2) we have

$$2\mu S=W(\mu\cos\alpha+\sin\alpha).$$

Hence, from (3),

$$AG=\frac{a}{\cos\alpha}\frac{S}{W}=\frac{a}{\cos\alpha}\frac{\mu\cos\alpha+\sin\alpha}{2\mu}$$

$$=\frac{a}{2}(1+\tan\alpha\cot\lambda)$$

Hence the least length $=2AG=a(1+\tan\alpha\cot\lambda)$.

4. Let AB be the rod of weight W, A being the rough end. Let G be the middle point of AB, and O be the centre of the hoop. The reaction at the smooth end B acts along the radius BO; if the reaction at A act along AE, the $\angle OAE=\lambda$. The directions of the reactions at A and B, and W, meet in E, G being vertically below E. Also GE is parallel to AO; therefore E bisects BO. Thus we have $OA=OB=10$ ins., $OE=EB=5$ ins., $AG=GB=6$ ins., and

$$OG=\sqrt{OB^2-BG^2}=8 \text{ ins.}; \text{ also } EG=\tfrac{1}{2}OA=5 \text{ ins.}=EB.$$

Also the $\angle AEG=$ the $\angle OAE=\lambda$.

Let the $\angle EGB = \alpha =$ the $\angle EBG$. Then

$$\frac{GE}{AG} = \frac{\sin(\alpha - \lambda)}{\sin\lambda} = \sin\alpha\cot\lambda - \cos\alpha.$$

$$\therefore \frac{5}{6} = \frac{4}{5}\cdot\cot\lambda - \frac{3}{5}, \text{ so that } 24\cot\lambda = 43.$$

$$\therefore \tan\lambda = \mu = \frac{24}{43}.$$

5. By equation (2) of Art. 209, we have

$$2\tan\theta = \tan(60^\circ + \lambda) - \tan(60^\circ - \lambda)$$

$$= \frac{\sqrt{3}+\mu}{1-\mu\sqrt{3}} - \frac{\sqrt{3}-\mu}{1+\mu\sqrt{3}} = \frac{8\mu}{1-3\mu^2};$$

$$\therefore \tan\theta = \frac{4\mu}{1-3\mu^2} = \frac{\frac{4}{\sqrt{3}}\tan\alpha}{1-\tan^2\alpha} = \frac{2}{\sqrt{3}}\cdot\frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2}{\sqrt{3}}\tan 2\alpha;$$

$$\therefore \tan\theta : \tan 2\alpha = 2 : \sqrt{3}.$$

6. By the position of limiting equilibrium is meant that A and B are on the point of sliding towards each other, since there could be equilibrium with A and B together without any friction. The tension T of the string is uniform throughout; therefore by resolving horizontally for the equilibrium of C it appears that AC and BC make equal angles, θ say, with the vertical, and the position is symmetrical. Hence, if R be the normal reaction at A or B, W be the weight of each ring, and $\mu = 3^{-\frac{3}{2}} = \frac{1}{3\sqrt{3}}$; we have for the equilibrium of the system, resolving vertically, $2R = 3W$, *i.e.*

$$R = \frac{3}{2}W \quad\text{..............................(1)}$$

for the equilibrium of C, resolving vertically, $2T\cos\theta = W$, *i.e.*

$$T = \frac{W}{2\cos\theta} \quad\text{..............................(2);}$$

for the equilibrium of A, resolving horizontally,

$$T\sin\theta = \mu R \quad\text{..............................(3).}$$

Substituting in (3) for R and T from (1) and (2), and for μ, we have

$$\frac{W}{2}\cdot\tan\theta = \frac{1}{3\sqrt{3}}\cdot\frac{3W}{2},$$

$$i.e. \ \tan\theta = \frac{1}{\sqrt{3}}, \text{ and } \theta = 30^\circ.$$

Thus the angle $ACB = 60^\circ$, and the triangle ABC is equilateral.

7. AC being fixed, the motion of A, if any, would be along AC; therefore the friction is along CA. The directions of the three forces W the weight of AB acting at G its middle point, T the tension of the string, and R the resultant reaction at A must meet in a point O. Then OA makes the angle λ with the vertical.

Since $$AB=2BG, \quad \tan OAB=\frac{1}{2}\tan OGB,$$

$$i.e. \quad \tan(\alpha-\lambda)=\frac{1}{2}\tan\alpha.$$

$$\therefore \frac{\tan\alpha-\tan\lambda}{1+\tan\alpha\tan\lambda}=\frac{1}{2}\tan\alpha.$$

$$\therefore 2\tan\alpha-2\tan\lambda=\tan\alpha+\tan^2\alpha\tan\lambda,$$

whence $$\mu=\tan\lambda=\frac{\tan\alpha}{2+\tan^2\alpha}.$$

8. Let AB be the movable rod of weight W, AC (vertical) and CB (at an angle α to the horizon) be the fixed rods, R be the normal (horizontal) reaction at A, S be the normal reaction at B, and $2a$ be the length of AB. Let AB be on the point of sliding down at A and up at B. Then resolving vertically and horizontally, we have

$$\mu R+S\cos\alpha=W+\mu S\sin\alpha \quad \text{.....(1)},$$

and $$R=S\sin\alpha+\mu S\cos\alpha \quad \text{.......(2)}.$$

Also, taking moments about A, we have

$$S\,.\,2a\cos(\alpha+\theta)=W\,.\,a\cos\theta+\mu S\,.\,2a\sin(\alpha+\theta) \quad \text{.........(3)}.$$

Eliminating R from (1) and (2), we have

$$\mu S\sin\alpha+\mu^2S\cos\alpha+S\cos\alpha=W+\mu S\sin\alpha,$$

$$\therefore W=(1+\mu^2)S\cos\alpha \quad \text{.......(4)}.$$

Eliminating W from (3) and (4), we have

$$(1+\mu^2)S\cos\alpha=\frac{2S\cos(\alpha+\theta)}{\cos\theta}-\frac{2\mu S\sin(\alpha+\theta)}{\cos\theta}.$$

$$\therefore (1+\mu^2)\cos\alpha=2(\cos\alpha-\sin\alpha\tan\theta)-2\mu(\sin\alpha+\cos\alpha\tan\theta).$$

$$\therefore 1+\mu^2=2-2\tan\alpha\tan\theta-2\mu\tan\alpha-2\mu\tan\theta.$$

$$\therefore 2(\mu+\tan\alpha)\tan\theta=1-2\mu\tan\alpha-\mu^2.$$

$$\therefore \tan\theta=\frac{1-2\mu\tan\alpha-\mu^2}{2(\mu+\tan\alpha)}.$$

Otherwise thus: Let λ be the angle of friction.

The directions of W and the resultant reactions at A and B must pass through a point O.

The resultant reaction AO at A makes an angle λ with the normal at A, *i.e.* the horizontal, so that $\angle AOG=90°-\lambda$.

The normal at B makes an angle α with the vertical, and hence

$$\angle BOG = \alpha + \lambda.$$

Theorem (1) of Art. 79 then gives

$$2 \cot QGA = \cot(\alpha + \lambda) - \cot(90^\circ - \lambda),$$

i.e.
$$2 \tan \theta = \frac{1 - \mu \tan \alpha}{\tan \alpha + \mu} - \mu = \frac{1 - 2\mu \tan \alpha - \mu^2}{\mu + \tan \alpha}.$$

9. Take the figure p. 260, with $\mu = \mu' = \tan \lambda$; and let the required distance $DB = x$.

Resolving horizontally and vertically, we have

$$\mu R = S \qquad \text{.................. (1)},$$

and
$$R + \mu S = W + P \qquad \text{.................. (2)}.$$

Also, taking moments about A, we have

$$W \cdot \frac{a}{2} \cos \theta + P(a - x) \cos \theta = S \cdot a \sin \theta + \mu S \cdot a \cos \theta \quad \text{.....(3)}.$$

From (1) and (2), $\quad R(1 + \mu^2) = P + W,$

$$\textit{i.e.} \quad R = \frac{P + W}{1 + \mu^2};$$

and
$$S = \frac{\mu}{1 + \mu^2}(P + W) = \frac{\tan \lambda}{1 + \tan^2 \lambda}(P + W) = \frac{1}{2} \sin 2\lambda \, (P + W).$$

Hence (3) becomes

$$W \cdot \frac{a}{2} \cos \theta + P(a - x) \cos \theta = \frac{a}{2} \sin 2\lambda \, (P + W)(\sin \theta + \tan \lambda \cos \theta),$$

$$\textit{i.e.} \quad W \cdot a + 2P(a - x) = a\,(P + W) \sin 2\lambda \, (\tan \theta + \tan \lambda),$$

$$\begin{aligned}
2Px &= a[W + 2P - (P + W) \sin 2\lambda \, (\tan \theta + \tan \lambda)], \\
&= a[W + 2P - (P + W) \sin 2\lambda \tan \theta - 2(P + W) \sin^2 \lambda], \\
&= a[W(1 - 2\sin^2 \lambda) + 2P(1 - \sin^2 \lambda) - (P + W) \sin 2\lambda \tan \theta], \\
&= a[W \cos 2\lambda + 2P \cos^2 \lambda - (P + W) \sin 2\lambda \tan \theta], \\
&= a \sin 2\lambda \, [W \cot 2\lambda + P \cot \lambda - (P + W) \tan \theta],
\end{aligned}$$

$$\therefore \quad x = \frac{W \cot 2\lambda + P \cot \lambda - (P + W) \tan \theta}{2P} \, a \sin 2\lambda.$$

10. Let θ be the inclination of the peg to the vertical, and let the rope make an angle λ with the normal to the peg. Then

$$\tan(\theta+\lambda)=\frac{4}{2}=2;$$

$$\therefore \tan\theta+\tan\lambda=2(1-\tan\theta\tan\lambda),$$

$$\text{i.e. } \tan\theta+\frac{4}{3}=2\left(1-\frac{4}{3}\tan\theta\right),$$

whence $$\tan\theta=\frac{2}{11}, \text{ i.e. } \theta=\tan^{-1}\frac{2}{11}.$$

11. Let AB represent the lid; it rests against the wall at an angle of 30°, since $AB=12$ ins., and the distance, AD, of A from the wall $=6$ ins.

Also let G be the centre of gravity of the lid, supposed uniform, and R be the horizontal reaction of the (smooth) wall. Then the resultant of the hinge acting at A must pass through the point where the vertical through G meets the direction of R, and make an angle θ, say, with the vertical; also, producing this line of action to meet the wall at C, we have $BC=BD$, by similar triangles.

Hence $$\tan\theta=\frac{AD}{DC}=\frac{1}{2}\tan ABD=\frac{1}{2}\tan 30^\circ=\frac{1}{2\sqrt{3}},$$

and, by Lami's Theorem,

$$R:50=\sin\theta:\cos\theta,$$

$$\therefore R=50\tan\theta=\frac{50}{2\sqrt{3}}.$$

Again, if S be the vertical reaction of the ground, resolving vertically and horizontally for the whole system, we have

$$S=250, \text{ and } \mu S=R;$$

$$\therefore \mu\times 250=\frac{50}{2\sqrt{3}}, \text{ whence } \mu=\frac{\sqrt{3}}{30}.$$

The point of action of S is somewhere within the base of the chest, to be determined by drawing, from the intersection of CA and the vertical through the centre of gravity of the chest, a line at an angle $\tan^{-1}\mu$ to the vertical.

12. Let W be the weight of the disc, R be the normal reaction of the plane, and T be the tension of the string, which must touch the disc at the other end of the diameter through the point of contact of the disc and the plane. Resolving along and perpendicular to the plane, we have

$$T+\mu R=W\sin i \quad\text{..........................(1)},$$

and $$R=W\cos i \quad\text{..........................(2)}.$$

Also, moments about the point of contact give, if a be the radius of the disc,

$$T \cdot 2a = W \cdot a \sin i \qquad (3).$$

Substituting for T and R from (2) and (3) in (1), we have

$$\frac{W}{2} \sin i + \mu W \cos i = W \sin i,$$

whence $\mu = \frac{1}{2} \tan i$, in the extreme case when the disc is on the point of slipping.

13. The vertical pressure of the table on the particle P upon it is equal to W its weight; also the tension of the second string is equal to W; hence if $\mu >$ or $= 1$, the particle will rest without any tension of the first string anywhere within the circle whose centre is A and whose radius is a. Also, if BC be the edge of the table and AC be perpendicular to it, then, BC being smooth, for the equilibrium of the suspended particle, PB must be perpendicular to the edge BC. Thus the distance of the piece of string PB from A $(= BC)$ is not $> a$.

If μ be < 1, there must be some tension in the string to maintain equilibrium.

If θ be the angle PAC we have, by taking moments about A to avoid the friction,

$$W \times a \sin \theta = \text{moment of the friction}$$

Now when the friction is limiting it has both its greatest value and also, since it then acts through P at right angles to AP, its distance from A is greatest; its moment is therefore then greatest and equal to $\mu W \times a$.

Hence the greatest value of θ is given by

$$W \cdot a \sin \theta = \mu W \times a.$$

The greatest distance of P from AC therefore $= a \sin \theta = \mu a$.

14. If W be the weight of the beam, and R and S be the normal reactions of the wall and the peg, then (1) resolving vertically and horizontally, and taking moments about the point of contact with the wall, we have

$$\mu R + S(\mu \cos \theta + \sin \theta) = W \qquad \text{(i)},$$

$$R = S(-\cos \theta + \mu \sin \theta) \qquad \text{(ii)},$$

$$S \cdot c \operatorname{cosec} \theta = W \cdot a \sin \theta \qquad \text{(iii)}.$$

From (i) and (ii), $\quad W = S(\mu^2 + 1) \sin \theta,$

so that from (iii),

$$c \operatorname{cosec} \theta = a \sin^2 \theta (\mu^2 + 1) = a \sin^2 \theta \sec^2 \lambda.$$

$$\therefore \sin^3 \theta = \frac{c}{a} \cos^2 \lambda.$$

(2) As before, $\mu R + S(\mu \cos\theta + \sin\theta) = W$;

also $R = S(\cos\theta - \mu \sin\theta)$.

$$\therefore\ W = S[2\mu\cos\theta + (1-\mu^2)\sin\theta];$$

and $S \cdot c \operatorname{cosec}\theta = W' \cdot a \sin\theta$;

hence

$$\frac{c}{a}\operatorname{cosec}^2\theta = \sec^2\lambda\,[2\sin\lambda\cos\lambda\cos\theta + (\cos^2\lambda - \sin^2\lambda)\sin\theta].$$

$$\therefore\ \frac{c}{a}\cos^2\lambda = \sin^2\theta\,(\sin 2\lambda\cos\theta + \cos 2\lambda\sin\theta),$$
$$= \sin^2\theta\sin(\theta + 2\lambda).$$

The third case follows by changing the sign of μ, and, therefore, of λ.

15. The sphere being smooth, for the equilibrium of the disc the particle must lie in that diameter of the disc which is in the vertical plane through the centres of the sphere and the disc; therefore the disc may be represented by the straight rod AB lying in a smooth hoop whose centre is C. Let G be the centre of AB, D be the position of the particle, and λ be the angle of friction. The reactions at A and B pass through C; therefore the resultant of W and w passes through C, and $DE : EG = W : w$, E being the point where the resultant cuts AB; therefore

$$DG : EG = W + w : w, \quad \text{and} \quad DG = EG\left(\frac{W}{w} + 1\right);$$

also $EG = OG\tan\lambda = \sqrt{b^2 - a^2}\,.\,\mu$;

hence $$DG = \mu\left(\frac{W}{w} + 1\right)\sqrt{b^2 - a^2}.$$

16. Let R be the normal reaction of the prism on the sphere, and S be the normal reaction of the horizontal plane on the prism. Then, for the sphere, resolving vertically, we have $W = R\cos\alpha$; for both sphere and prism, resolving vertically, we have $W + W' = S$, where W' is the weight of the prism; for the prism, resolving horizontally, we have $\mu S = R\sin\alpha$. Hence

$$\mu(W + W') = W\tan\alpha, \quad \text{and} \quad W' = W\left(\frac{\tan\alpha}{\mu} - 1\right).$$

If W' be less than this, there will not be sufficient friction.

17. If A and C be the middle points of the rods, fastened at B, then G, their centre of gravity, is the middle point of AC; and if GD, parallel to CB, meet AB in D, then if the peg were smooth the system would rest with the peg at D, and AB horizontal; and the limiting positions of the peg are E and F, on AB, such that GE and GF make angles $\tan^{-1}\mu$ with the perpendiculars to AB at those points; in the two cases GE and GF respectively being vertical, so

that when the peg is at E the point of contact is just about to slip in the direction AB, and when the peg is at F it is just about to slip in the direction BA.

Also $\qquad AD=\frac{a}{2}=GD$; hence $AE=\frac{a}{2}-\frac{a}{2}\mu$,

and $$AF=\frac{a}{2}+\frac{a}{2}\mu;$$

hence the limiting distances of the points of contact from A are

$$\frac{a}{2}(1\mp\mu).$$

18. If D and E be the middle points of AC and BC respectively, for equilibrium DE, in which lies G the centre of gravity, must be vertical, and the $\angle DCE=\frac{\pi}{2}$; therefore, if AC make an $\angle\,\theta$ with the horizon, we have

$$DC\cos\theta=CE\sin\theta, \quad i.e. \quad DC\cos\theta=3DC\sin\theta,$$

and, therefore, $\tan\theta=\frac{1}{3}$.

Again, when A is on the edge of the table, the normal reaction R is perpendicular to AC, and if AC make an $\angle\,\alpha$ with the vertical, we have

$$\mu R\sin\alpha=R\cos\alpha, \quad i.e. \ \cot\alpha=\mu \ldots\ldots\ldots\ldots\ldots(1),$$

and $$\mu R\cos\alpha+R\sin\alpha=4W\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where W and $3W$ are the weights of AC and BC respectively. Also moments about E give

$$\mu R\,.\,\frac{BC}{2}=R\,.\,AC+W\left(CE\cos\alpha-\frac{AC}{2}\sin\alpha\right),$$

i.e. $$\mu R\,.\,\frac{3}{2}-R=\frac{W}{2}(3\cos\alpha-\sin\alpha).$$

$$\therefore\ 4\left(\frac{3}{2}\mu-1\right)=(\mu\cos\alpha+\sin\alpha)\,\frac{3\cos\alpha-\sin\alpha}{2},\ \text{by (2)}.$$

$$\therefore\ 2(3\cos\alpha-\sin\alpha)=\frac{3\cos\alpha-\sin\alpha}{2}, \quad \text{whence} \quad \tan\alpha=\frac{9}{7}.$$

Hence, by (1), $$\mu=\cot\alpha=\frac{7}{9}.$$

19. Let C be the common vertex, ACB be the string, α and β be the inclinations to the horizon of the planes on which lie the portions AC and CB respectively, W and W' be the weights of AC and CB acting at D and E their middle points respectively, R and S be the resultant normal reactions of the planes at D and E, T be the tension of the string round the pulley (supposed smooth), the same on both sides, and λ be the angle of friction. Suppose A to be on the point of descending. The line joining A and B is parallel to the line

joining D and E, and if it makes an angle θ with the horizon, the $\angle CDE=\alpha-\theta$, and the $\angle CED=\beta+\theta$. Also we have

$$W : W'=CD : CE=\sin(\beta+\theta) : \sin(\alpha-\theta).$$

By Lami's Theorem,

$$\frac{T}{W}=\frac{\sin(\alpha-\lambda)}{\sin\left(\frac{\pi}{2}-\lambda\right)}, \text{ and } \frac{T}{W'}=\frac{\sin(\beta+\lambda)}{\sin\left(\frac{\pi}{2}+\lambda\right)}.$$

Hence, by division,

$$\frac{W'}{W}=\frac{\sin(\alpha-\lambda)}{\cos\lambda}\cdot\frac{\cos\lambda}{\sin(\beta+\lambda)};$$

hence
$$\frac{\sin(\alpha-\theta)}{\sin(\beta+\theta)}=\frac{\sin(\alpha-\lambda)}{\sin(\beta+\lambda)}.$$

$$\therefore\ \theta=\lambda.$$

Otherwise thus: Since μR acts upwards, μ being the coefficient of friction, we have

$$W\sin\alpha=\mu R+T, \text{ and } W\cos\alpha=R;$$

whence
$$T=W(\sin\alpha-\mu\cos\alpha).$$

So on the other plane, where μS acts downwards, we have

$$-T=W'(\sin\beta+\mu\cos\beta).$$

Therefore
$$W(\sin\alpha-\mu\cos\alpha)=W'(\sin\beta+\mu\cos\beta);$$

but $W : W'=l : l'$, where l and l' are the lengths of the portions AC and CB respectively, and $\mu=\tan\lambda$; hence we have

$$l\sin\alpha-l\tan\lambda\cos\alpha=l'\sin\beta+l'\tan\lambda\cos\beta,$$

i.e.
$$l\sin(\alpha-\lambda)=l'\sin(\beta+\lambda).$$

But
$$l : l'=\sin ABC : \sin CAB,$$
$$=\sin(\beta+\theta) : \sin(\alpha-\theta);$$

thus
$$\frac{\sin(\beta+\lambda)}{\sin(\alpha-\lambda)}=\frac{l}{l'}=\frac{\sin(\beta+\theta)}{\sin(\alpha-\theta)};$$

hence $\theta=\lambda$.

20. If α be the inclination of the plane to the horizon, and R be the normal reaction, we have

$$R=W\cos\alpha, \text{ and } \mu R+\frac{W}{2}=W\sin\alpha;$$

also
$$\mu=\frac{1}{2};$$

$$\therefore\ \frac{W}{2}\cos\alpha+\frac{W}{2}=W\sin\alpha, \quad i.e.\ 2\sin\alpha-1=\cos\alpha;$$

$$\therefore\ 4\sin^2\alpha-4\sin\alpha+1=1-\sin^2\alpha, \text{ whence } \sin\alpha=\frac{4}{5},$$

and
$$\cos\alpha=\frac{3}{5}.$$

In the second case let P be the required force and θ be the angle it makes with the line of greatest slope.

Since the resultant of P and μR is to balance the resultant of $\frac{W}{2}$ and $W \sin \alpha$, P will be least when it acts in the same direction as μR.

Hence we have

$$(P+\mu R)\cos\theta + \frac{W}{2}\cos 60^\circ = W\sin\alpha,$$

and
$$(P+\mu R)\sin\theta = \frac{W}{2}\sin 60^\circ;$$

$$\therefore \left(P+\frac{1}{2}W\cos\alpha\right)\cos\theta = W\sin\alpha - \frac{W\cos 60^\circ}{2},$$

i.e.
$$\left(P+\frac{3W}{10}\right)\cos\theta = W\left(\frac{4}{5}-\frac{1}{4}\right) = \frac{11W}{20} \text{ .... (i)},$$

and
$$\left(P+\frac{3W}{10}\right)\sin\theta = \frac{W\sqrt{3}}{4} \text{ .... (ii)}$$

Hence, squaring and adding (i) and (ii), we have

$$\left(P+\frac{3W}{10}\right)^2 = W^2\left(\frac{121}{400}+\frac{3}{16}\right) = \frac{196W^2}{400};$$

$$\therefore P+\frac{3W}{10} = \frac{14W}{20} = \frac{7W}{10}, \text{ and } P = \frac{2W}{5};$$

also, from (i),

$$\frac{7W}{10}\cos\theta = \frac{11W}{20}, \text{ so that } \cos\theta = \frac{11}{14},$$

i.e.
$$\theta = \cos^{-1}\frac{11}{14}.$$

21. If AB be a line of greatest slope in the plane and W be at C, its components down and perpendicular to the plane are each $\frac{W}{\sqrt{2}}$, and the tension of the string CA is equal to P; also if W be on the point of moving along CB, where the angle $ABC=\phi$ say, the friction acts along BC. Hence, resolving horizontally and along the line of greatest slope, we have

$$P\sin\theta = \mu\frac{W}{\sqrt{2}}\sin\phi,$$

and
$$\frac{W}{\sqrt{2}} = P\cos\theta + \mu\frac{W}{\sqrt{2}}\cos\phi;$$

i.e. $\sqrt{2}\,.\,\sin\theta = \sqrt{3}\,.\,\sin\phi$, and $3-\sqrt{2}\,.\,\cos\theta = \sqrt{3}\,.\,\cos\phi$;
squaring and adding these last two equations, we have

$$9 - 6\sqrt{2}\cos\theta + 2 = 3, \text{ whence } \cos\theta = \frac{8}{6\sqrt{2}} = \frac{2\sqrt{2}}{3}.$$

Also $$\cos\phi = \frac{3-\sqrt{2}\cos\theta}{\sqrt{3}} = \frac{3-\frac{4}{3}}{\sqrt{3}} = \frac{5}{3\sqrt{3}} = \frac{5}{9}\sqrt{3},$$

i.e. $$\phi = \cos^{-1}\frac{5}{9}\sqrt{3}.$$

22. Let R be the normal reaction, θ be the angle the friction makes with the line of greatest slope, and P be the required force. The friction being opposed to the motion, the body will begin to move in the contrary direction to the friction. Resolving horizontally, we have $P=\mu R\sin\theta$; resolving along the line of greatest slope, we have $W\sin\alpha=\mu R\cos\theta$; and resolving perpendicular to the plane, we have $R=W\cos\alpha$. Hence we have

$$P=\mu W\sin\theta\cos\alpha=2W\sin\theta\sin\alpha,$$

since $$\mu=2\tan\alpha;$$

and $$W\sin\alpha=\mu W\cos\alpha\cos\theta,$$

i.e. $$\sin\alpha=2\sin\alpha\cos\theta;$$

$$\therefore \cos\theta=\frac{1}{2}; \quad \textit{i.e. } \theta=60^\circ;$$

and $$P=2W\sin 60^\circ\sin\alpha=\sqrt{3}\,.\,W\sin\alpha.$$

23. Let the rod make an angle θ with the line of greatest slope, the frictions make angles ϕ_1 and ϕ_2 with the same line, T be the tension of the rod, and W be each weight. The reaction at right angles to the plane is $W\cos\alpha$ in each case $=R$; also

$$\tan\alpha=\sqrt{\mu_1\mu_2};$$

$$\therefore W\sin\alpha=R\tan\alpha=R\sqrt{\mu_1\mu_2}.$$

Resolving along the line of greatest slope, we have

$$W\sin\alpha+T\cos\theta=\mu_1 R\cos\phi_1,$$

and $$W\sin\alpha-T\cos\theta=\mu_2 R\cos\phi_2;$$

resolving horizontally, we have

$$T\sin\theta=\mu_1 R\sin\phi_1,$$

and $$T\sin\theta=\mu_2 R\sin\phi_2.$$

Eliminate ϕ_1 and ϕ_2; thus we have

$$(W\sin\alpha+T\cos\theta)^2+(T\sin\theta)^2=(\mu_1 R)^2,$$

and $$(W\sin\alpha-T\cos\theta)^2+(T\sin\theta)^2=(\mu_2 R)^2;$$

or $$\mu_1\mu_2 R^2+2RT\sqrt{\mu_1\mu_2}\cos\theta+T^2=\mu_1{}^2R^2,$$

and $$\mu_1\mu_2 R^2 - 2RT\sqrt{\mu_1\mu_2}\cos\theta + T^2 = \mu_2^2 R^2;$$

hence, by addition, we have

$$2\mu_1\mu_2 R^2 + 2T^2 = (\mu_1^2 + \mu_2^2) R^2.$$

$$\therefore\ 2T^2 = (\mu_1^2 - 2\mu_1\mu_2 + \mu_2^2) R^2.$$

$$\therefore\ \sqrt{2}\,.\,T = (\mu_1 - \mu_2) R;$$

also, by subtraction, we have

$$4RT\sqrt{\mu_1\mu_2}\cos\theta = (\mu_1^2 - \mu_2^2) R^2;$$

hence, by division, we have

$$2\sqrt{2\mu_1\mu_2}\cos\theta = \mu_1 + \mu_2;$$

$$\therefore\ \cos\theta = \frac{\mu_1 + \mu_2}{2\sqrt{2\mu_1\mu_2}},\quad i.e.\ \ \theta = \cos^{-1}\left(\frac{\mu_1 + \mu_2}{2\sqrt{2\mu_1\mu_2}}\right).$$

24. The particle, of weight W, can only move round A; therefore the friction $\mu W \cos\alpha$ acts perpendicular to AP. Hence, resolving at right angles to AP on the plane, we have

$$\mu W \cos\alpha = W \sin\alpha \sin\theta;$$

$$\therefore\ \sin\theta = \mu \cot\alpha.$$

If $\mu\cot\alpha > 1$, the whole possible friction is not required, the coefficient brought into play being only $\sin\theta\,.\tan\alpha$ which is $<\tan\alpha$, and therefore $<\mu$; hence the particle will rest for any value of θ. Moreover, in this case, since $\mu > \tan\alpha$, the particle will rest without any tension of the string, *i.e.* the string may be removed.

25. Let C be the centre of the base of the shell, A be the point of contact of the shell with the plane, and G be the centre of gravity of the shell. Then, for equilibrium, AG is vertical, and therefore if α be the inclination of the plane to the horizon we have $\angle CAG = \alpha$, since CA is perpendicular to the plane.

Also the plane base is inclined to the horizon at the angle supplementary to the angle CGA, $= \theta$ say. Then we have

$$\frac{CA}{CG} = \frac{\sin\theta}{\sin\alpha};$$

but $CE = 2CG$, where CE is the radius perpendicular to the plane base [Art. 120];

$$\therefore\ CA = CE = 2CG;$$

hence $$\sin\theta = 2\sin\alpha;$$

and the greatest value of α is the angle of friction λ; therefore the greatest value of θ is $\sin^{-1}(2\sin\lambda)$.

26. Let E and F be the points of contact of the hemisphere with the plane and the wall respectively, C be the centre of the base of the hemisphere, CD be the radius perpendicular to its plane face, and G be its centre of gravity, so that

$$CG=\frac{3}{8}CD=\frac{3}{8}r, \text{ say.}$$

If the vertical through G meet the horizontal radius FC (along which the normal reaction at F acts) in O, the reaction at E must pass along EO; therefore the angle $CEO=\lambda$, where $\tan\lambda=\mu$. Also, if θ be the angle that the base of the hemisphere in its limiting position of equilibrium makes with the vertical,

$$CO=CG\cos\theta=\frac{3}{8}r\cos\theta;$$

also the greatest value of CO is when CEO is λ, and therefore

$$=CE\tan\lambda=r\tan\lambda.$$

The greatest value of $\frac{3r}{8}\cos\theta$ is therefore $r\tan\lambda$, so that the least value of θ is

$$\cos^{-1}\left(\frac{8}{3}\mu\right).$$

So long as $\mu<\frac{3}{8}$ this always gives a real value.

If however $\mu>\frac{3}{8}$ this limiting value of θ is impossible, *i.e.* there is no limiting value for θ, *i.e.* the hemisphere will rest in any position.

In the general case let R, S be the normal reactions at E and F and therefore μR and $\mu' S$ the frictions towards the wall and vertically upwards respectively.

Then, resolving horizontally and vertically, we have

$$S=\mu R \quad\text{..........................}\quad (1),$$

$$R+\mu' S=W \quad\text{..........................}\quad (2).$$

Also, taking moments about C, we have

$$(\mu R+\mu' S)\,a=W.\frac{3a}{8}\cos\theta \quad\text{.....................}\quad (3),$$

where θ is the inclination of the base to the vertical.

Solving (1) and (2), we have

$$\frac{R}{1}=\frac{S}{\mu}=\frac{W}{1+\mu\mu'}.$$

Hence (3) gives

$$\frac{W}{1+\mu\mu'}(\mu+\mu\mu')=W.\frac{3}{8}\cos\theta, \text{ i.e. } \cos\theta=\frac{8\mu}{3}\cdot\frac{1+\mu'}{1+\mu\mu'}.$$

27. Let C be the centre of the base of the hemisphere, A be its point of contact with the plane, G be its centre of gravity, and α be the inclination of the plane to the horizon. Then, for equilibrium, AG is vertical; also

$$CG=\frac{3}{8}CA.$$

[Art. 117.]

Hence we have

$$\sin\alpha=\sin CAG=\frac{3}{8}\sin CGA;$$

thus $\sin\alpha$ is seen to increase with $\sin CGA$, and since the greatest value of $\sin CGA$ is 1, the greatest value of $\sin\alpha$ is $\frac{3}{8}$, *i.e.*

$$\alpha=\sin^{-1}\frac{3}{8}.$$

If there be equilibrium the resultant reaction of the plane must balance the weight of the sphere.

But this reaction passes through the point of contact of the sphere and plane. Also the weight acts vertically through the centre. Hence unless the supporting plane be horizontal the weight cannot pass through the point of contact, as is clear from a figure. Hence there cannot be equilibrium.

28. Let C be the centre of the base of the hemisphere, A be its point of contact with the plane, G be its centre of gravity and $90°-\theta$ be the inclination of the plane base to the vertical.

Then for equilibrium, AG is vertical, and if

$$\alpha=\sin^{-1}\frac{3}{16},$$

the $\angle CAG=\alpha$. Also, by Art. 117,

$$CG=\frac{3}{8}CA.$$

Hence we have
$$\frac{\sin\alpha}{\sin\theta}=\frac{\sin CAG}{\sin CGA}=\frac{CG}{CA}=\frac{3}{8};$$

$$\therefore \sin\theta=\frac{8}{3}\sin\alpha=\frac{8}{3}\cdot\frac{3}{16}=\frac{1}{2},\ \textit{i.e. } \theta=30°;$$

and the required angle $=90°-30°=60°$.

29. Draw a figure representing the vertical section through P the position of the particle and C the centre of the base of the hemisphere. Let A be the point of contact of the plane and the hemisphere, G be the centre of gravity of the hemisphere, and let the base of the hemisphere make an angle θ with the vertical. Then, since the horizontal plane produces only a vertical reaction at A (there

being no external horizontal force), AC is vertical if the moments about C of W and W' be equal, *i.e.* if

$$W' . CP \sin\theta = W . CG \cos\theta,$$

$$i.e.\quad CP = \frac{W}{W'} . \frac{3}{8} a . \cot\theta;$$

and for the equilibrium of W' on the inclined plane CP, the greatest value of $\frac{\pi}{2} - \theta$ is $\tan^{-1}\mu$; hence the greatest value of $\tan\left(\frac{\pi}{2} - \theta\right)$, *i.e.* of $\cot\theta$, is μ; and therefore the greatest value of CP is $\frac{3W\mu a}{8W'}$.

30. Let O be the centre of the sphere. For limiting equilibrium the centre of gravity of the sphere must lie in the vertical through A its point of contact with the plane, which will make an angle α with OA the radius to A; and if AB be the vertical chord of the sphere through A, the two points G_1 and G_2 in BA, at distance c from O, are the two limiting positions of the centre of gravity of the sphere; and if the angle $G_1OG_2 = 2\theta$, 2θ is the angle through which the sphere may be turned from one limiting position of equilibrium to the other. In intermediate positions of equilibrium, the centre of gravity will lie on the arc G_1G_2 of the circle centre O and radius c in the plane AOB. Also, if ON be perpendicular to AB, we have

$$ON = a \sin\alpha = c \cos\theta;$$

hence
$$2\theta = 2\cos^{-1}\left(\frac{a \sin\alpha}{c}\right).$$

EXAMPLES. XXXVI. (Pages 277, 278.)

1. If X be the required force at the point P in BC, then the beam BC is in equilibrium under the action of the vertical forces X and W its weight, and R the action at B; therefore this action must be vertical. For the equilibrium of the beam AB, moments about A shew that the action R on AB is upwards, and give

$$R \cdot AB = W \cdot \frac{AB}{2}, \text{ i.e. } R = \frac{W}{2}.$$

Hence, from the equilibrium of the beam BC,

$$X = R + W = \frac{3W}{2};$$

also moments about B give

$$X \cdot BP = W \cdot \frac{BC}{2},$$

so that $$\frac{3W}{2} \cdot BP = W \cdot \frac{BC}{2}, \text{ i.e. } BP = \frac{1}{3}BC.$$

2. Since the beams are each inclined at an angle of 60° to the horizon, ABC is equilateral, and therefore the beams each weigh 30 lbs. From the symmetry of the system, the actions of the hinge at C must be equally inclined to the beams in the same sense, and, therefore, being equal and opposite ($=R$, say), these actions must be horizontal. Consider the beam AC; taking moments about A, we have

$$R \cdot AC \sin 60° = 30 \cdot \frac{AC}{2} \cos 60°;$$

hence $$R = 15 \cot 60° = 5\sqrt{3} \text{ lbs. wt.}$$

3. The pegs being in the same horizontal line and at the middle points of the legs, the position is symmetrical; therefore the thrust T in the weightless rod and the action S at the hinge are both horizontal. Hence, resolving for the equilibrium of either leg along its length, we have

$$W \cos \alpha = (T+S) \sin \alpha;$$

also, moments about the centre of the leg give $T=S$;

$$\therefore\ T = S = \frac{W \cot \alpha}{2}.$$

4. Let D be the middle point of AB, R be the normal reaction at B or C, T be the tension of the string, $2a$ be the length of either rod, and the $\angle DCB$ be denoted by θ. Also, let Y and X be the vertical and horizontal components respectively of the action at A. For

the system, resolving vertically, we have $2R=2W$, *i.e.* $R=W$. Also, if DC meet AE, the perpendicular from A on BC, in F,

$$\tan\theta=\frac{FE}{CE}=\frac{\frac{1}{3}AE}{CE}=\frac{1}{3}\tan\alpha.$$

For either rod, resolving horizontally and vertically, we have

$$X=T\cos\theta,\quad\text{and}\quad Y=T\sin\theta;$$

hence the action at A

$$=\sqrt{X^2+Y^2}=T,$$

acting at an angle $\tan^{-1}\frac{Y}{X}$, *i.e.* θ, to the horizon. [This is obvious; for, since $R=W$, T must be equal and in the opposite direction to the action at A.] For AB, taking moments about D, we have

$$R\,.\,a\cos\alpha=X\,.\,a\sin\alpha+Y\,.\,a\cos\alpha;$$

$$\therefore\ W-X\tan\alpha=Y.$$

$$\therefore\ W-T\tan\alpha\cos\theta=T\sin\theta.$$

$$\therefore\ W\sec\theta=T(\tan\theta+\tan\alpha).$$

$$\therefore\ T\,.\,\frac{4}{3}\tan\alpha=W\sqrt{1+\tan^2\theta}=W\sqrt{1+\frac{1}{9}\tan^2\alpha}.$$

$$\therefore\ 4T\sin\alpha=W\sqrt{9\cos^2\alpha+\sin^2\alpha}.$$

$$\therefore\ T=\frac{W}{4}\operatorname{cosec}\alpha\sqrt{1+8\cos^2\alpha}.$$

5. By symmetry, the thrust at C is horizontal, and if θ be the inclination of AC to the horizon, and μ be the coefficient of friction actually employed then, S be the thrust at C, and R be the normal reaction at A, for the equilibrium of AC, we have $S=\mu R$, and $R=W$, the weight of AC; also moments about A give

$$S\,.\sin\theta=\frac{W}{2}\,.\cos\theta.$$

Hence $$\mu\sin\theta=\frac{1}{2}\cos\theta,$$

so that $$\tan\theta=\frac{1}{2\mu};$$

therefore when μ is greatest, θ is least, but the $\angle ACB$ $(=\pi-2\theta)$ is greatest; therefore if $\mu=\frac{1}{2}$, in the limit

$$\text{the }\angle ACB=\pi-2\tan^{-1}1=\pi-\frac{\pi}{2}=\frac{\pi}{2}$$

at most. Also

$$S=\frac{W}{2}\cot\theta=\frac{W}{2}\tan\left(\frac{1}{2}ACB\right).$$

6. If the weights of the rods be $5w$, $4w$ and $3w$ respectively, then $W=12w$. Also, since $AB^2=BC^2+CA^2$, the angle ACB is a right angle; and if F be the fulcrum, at distance x from D the middle point of AB, moments about F for the equilibrium of the whole give

$$5w\,.\,x+4w\left(\frac{5}{2}+x-2\cos ABC\right)=3w\left(\frac{5}{2}-x-\frac{3}{2}\sin ABC\right).$$

$$\therefore\ 5x+4\left(\frac{5}{2}+x-2\,.\,\frac{4}{5}\right)=3\left(\frac{5}{2}-x-\frac{3}{2}\cdot\frac{3}{5}\right),$$

whence $$x=\frac{1}{10}\text{ foot}=1\tfrac{1}{5}\text{ in.};$$

$$\therefore\ FA=30-1\tfrac{1}{5}=28\tfrac{4}{5}\text{ ins., and } FB=31\tfrac{1}{5}\text{ ins.}$$

Again, if P and Q be the vertical components of the actions at the hinges A and B respectively, then

$$P+Q+5w=W,\ \textit{i.e.}\ P+Q=12w-5w=7w;$$

also P and Q with the weight of AB balance about F; hence we have

$$P\times 28\tfrac{4}{5}=5w\times 1\tfrac{1}{5}+Q\times 31\tfrac{1}{5}.$$

$$\therefore\ 144P=30w+156Q=30w+156\,(7w-P),$$

whence $$P=\frac{187}{50}\,w=\frac{187}{600}\,W;$$

and $$Q=\left(7-\frac{187}{50}\right)w=\frac{163}{50}\,w=\frac{163}{600}\,W.$$

Otherwise thus: Taking moments about B for the whole, we have

$$4w\,.\,2\cos ABC+5w\,.\,\frac{5}{2}+3w\,.\left(5-\frac{3}{2}\cos BAC\right)=12w\,.\,BF.$$

$$\therefore\ 12BF=8\,.\,\frac{4}{5}+\frac{25}{2}+3\left(5-\frac{3}{2}\cdot\frac{3}{5}\right),$$

whence $$BF=\frac{312}{120}\text{ ft.}=2\tfrac{1}{2}\text{ ft.}+\frac{1}{10}\text{ ft.};$$

$$\therefore\ DF=\frac{1}{10}\text{ ft.}=1\tfrac{1}{5}\text{ in., towards } A.$$

Again, as before, $P+Q=7w$; and moments about B give

$$5w\,.\,\frac{5}{2}+5P=12w\,.\,\frac{312}{120}=\frac{312w}{10};$$

whence P and Q, as before.

7. Let D and E be the middle points of AB and BC respectively, W be the weight of each rod, T be the tension of the connecting string, and θ be the angle AB makes with the vertical.

The tension S of the suspending string at A = the weight of the whole $= 2W$. Also G, the centre of gravity of the system, is the middle point of DE, and AG is vertical; hence, if GH be perpendicular to AB, we have

$$\tan BAG = \frac{GH}{AH} = \frac{1}{2}BE \div \frac{3}{4}AB = \frac{1}{3},$$

i.e. $$\theta = \tan^{-1}\frac{1}{3}.$$

Hence BC makes an $\angle \tan^{-1}\frac{1}{3}$ with the horizon, and the connecting string makes equal angles of 45° with AB and BC. For the equilibrium of AB, taking moments about B, we have, if $2a$ be the length of a rod,

$$T \,.\, a \sin 45^\circ + W \,.\, a \sin \theta = 2W \,.\, 2a \sin \theta,$$

$$\therefore \frac{T}{\sqrt{2}} = 3W \sin \theta = W\frac{3}{\sqrt{10}}, \quad i.e. \quad T = \frac{3W}{\sqrt{5}}.$$

For BC, if Y and X be the vertical and horizontal components respectively of the action at B, resolving vertically and horizontally, we have

$$Y + W = T \sin (45^\circ + \theta) = \frac{T}{\sqrt{2}}(\cos\theta + \sin\theta),$$

and $$X = T \cos (45^\circ + \theta) = \frac{T}{\sqrt{2}}(\cos\theta - \sin\theta);$$

$$\therefore Y = \frac{3W}{\sqrt{10}} \cdot \frac{3+1}{\sqrt{10}} - W = \frac{2W}{10} = \frac{W}{5},$$

and $$X = \frac{3W}{\sqrt{10}} \cdot \frac{3-1}{10} = \frac{6W}{10} = \frac{3W}{5};$$

hence the action

$$= \sqrt{X^2 + Y^2} = \frac{W}{5}\sqrt{1+9} = \frac{W}{5}\sqrt{10},$$

acting at an angle

$$\tan^{-1}\frac{Y}{X}\left(i.e. \ \tan^{-1}\frac{1}{3}\right)$$

to the horizon.

8. Let T be the equal tension in the strings OA and OC, T' be the tension in the string OB, and R be the action at the hinge B. The triangles OAB and OBC are equilateral, and the angles are each 60°. For the system, resolving vertically, we have

$$T' + 2T \cos 60^\circ = 2W,$$

so that $$T' + T = 2W \qquad . \ (1).$$

For AB, taking moments about B, we have

$$T\,.\,12\sin 60^\circ = W\,.\,6\sin 60^\circ.$$

$$\therefore\ T=\frac{W}{2}.$$

Hence from (1),

$$T'=2W-\frac{W}{2}=\frac{3W}{2}.$$

Also, resolving horizontally, we have

$$R=T\cos 30^\circ,\ \ i.e.\ \ R=\frac{W}{2}\cdot\frac{\sqrt{3}}{2}=\frac{W\sqrt{3}}{4}.$$

9. Let P be the vertical pressure at the middle point of BC, R be the normal pressure at A or D, and O be the centre of the sphere.

Then, for equilibrium, $ABCD$ is in a vertical plane through O; and $AB=\frac{1}{2}AO$, so that

$$\angle AOB=\tan^{-1}\frac{1}{2}=\theta,\ \text{say};$$

thus AO makes an $\angle 2\theta$ with the vertical, and if W, $2W$ and W be the weights of the beams respectively, for AB, moments about B give

$$R\,.\,l=W\,.\,\frac{l}{2}\,.\cos 2\theta,$$

since AB is perpendicular to AO,

i.e. $$R=\frac{W\cos 2\theta}{2}\ \ldots\ldots\ldots\ldots\ldots\ (1).$$

Also, resolving vertically for the whole, we have

$$4W=P+2R\cos 2\theta;$$

hence, from (1), $$P=4W-W\cos^2 2\theta;$$

now $$\cos\theta=\frac{2}{\sqrt{5}}.$$

$$\therefore\ \cos 2\theta=2\cos^2\theta-1=\frac{8}{5}-1=\frac{3}{5};$$

$$\therefore\ P=W\left(4-\frac{9}{25}\right)=4W\times\frac{91}{100}.$$

10. Let the weights of the rods be aw, bw and cw, and R and S be the vertical actions on BC of the joints at B and C respectively.

For the equilibrium of BC, $R+S=bw$, and R must $=S$,

i.e. $$R=S=\frac{bw}{2},$$

i.e. the action at each joint = half the weight of the middle rod. For the equilibrium of AB, the pressure of the peg P, vertically upwards,

$$=aw+R=aw+\frac{bw}{2};$$

hence moments about B give

$$aw\,.\,\frac{a}{2}=BP\left(aw+\frac{bw}{2}\right);$$

i.e. $$BP=\frac{a^2}{2a+b};$$

also, resolving horizontally, the horizontal action at $B=0$. Similarly,

$$CQ=\frac{c^2}{2c+b},$$

interchanging a and c;

$$\therefore\ PQ=\frac{a^2}{2a+b}+\frac{c^2}{2c+b}+b.$$

Of course if the pegs were within BC, the outer rods would swing round into a vertical position.

Otherwise thus: $R=S=\frac{bw}{2}$, as before.

For AB, taking moments about P, we have

$$R\,.\,BP=aw\left(\frac{a}{2}-BP\right),$$

i.e. $$\frac{bw}{2}\,.\,BP=aw\left(\frac{a}{2}-BP\right),$$

whence $$BP=\frac{a^2}{2a+b}.$$

For CD, taking moments about Q, we have

$$S\,.\,CQ=cw\left(\frac{c}{2}-CQ\right),$$

i.e. $$\frac{bw}{2}\,.\,CQ=cw\left(\frac{c}{2}-CQ\right),$$

whence $$CQ=\frac{c^2}{2c+b}.$$

11. Since the weights and lengths of AB and AC are the same, for the equilibrium of the system the common centre of gravity must be vertically below A, and the rods make equal angles θ with the vertical; also the bar BD being in equilibrium under some force at B, and the reaction of AC at D which is perpendicular to AC, (the ring

being smooth), these forces must be in the same straight line, and therefore act along BD; thus the $\angle BDA$ is a right angle, and

$$\sin 2\theta = \frac{BD}{AB} = \frac{b}{a};$$

$$\therefore\ 2a \sin\theta \cos\theta = b(\sin^2\theta + \cos^2\theta),$$

$$\therefore\ b\tan^2\theta - 2a\tan\theta + b = 0,$$

whence $$\tan\theta = \frac{a \pm \sqrt{a^2 - b^2}}{b}.$$

Now $2\theta < \frac{\pi}{2}$, so that $\theta < \frac{\pi}{4}$, and $\tan\theta < 1$; but since b must be $< a$, or else equilibrium is impossible (cf. the value of $\tan\theta$), the upper sign of the radical would make $\tan\theta > \frac{a}{b}$, and, therefore, $\tan\theta > 1$: hence the lower sign only is applicable, and then

$$\tan\theta = \frac{a - \sqrt{a^2 - b^2}}{b} = \frac{a^2 - (a^2 - b^2)}{b[a + \sqrt{a^2 - b^2}]},$$

$$\theta = \tan^{-1}\frac{b}{a + \sqrt{a^2 - b^2}}.$$

12. Let $\frac{W}{4}$ be the weight, and $2a$ be the length of each rod; T be the tension of the connecting string, and Y and X be the vertical and horizontal components respectively of the action at B. For the equilibrium of AB, taking moments about A, we have

$$X \cdot 2a \sin 45° + Y \cdot 2a \sin 45° = \frac{W}{4} \cdot a \sin 45°.$$

$$\therefore\ X + Y = \frac{W}{8} \quad \text{(1)}$$

For the equilibrium of BC, resolving vertically, we have

$$\frac{T}{2} = Y + \frac{W}{4} \quad \text{(2)},$$

and taking moments about C, we have

$$\frac{W}{4} \cdot \frac{a}{\sqrt{2}} + Y \cdot \frac{2a}{\sqrt{2}} = X \cdot \frac{2a}{\sqrt{2}}.$$

$$\therefore\ X - Y = \frac{W}{8} \quad \text{(3)}.$$

From (1) and (3), $Y = 0$, and $X = \frac{W}{8}$, *i.e.* the action at B or D is one-eighth of the total weight of the rods acting in a horizontal direction.

Also, from (2), $$T = \frac{W}{2}.$$

13. Let $ABCD$ be the rhombus, T_1 be the tension of the string AC, and T_2 be the tension of the string BD. Let O be the point of intersection of AC and BD, which are at right angles to one another. The rod AB is in equilibrium under the actions upon it of the joints at A and B; these actions must therefore be equal (P, say) and opposite, and both act along AB. Similarly the action of the joint A on the rod AD is P along AD. Thus at A we have three forces T_1, P and P along AC, BA and DA, respectively, parallel to the sides of the triangle ACD; therefore $T_1 : P = AC : CD$. So at B we have three forces T_2, P and P along BD, AB and BC, respectively, parallel to the sides of the triangle BDC; therefore

$$T_2 : P = BD : CD;$$

hence
$$T_1 : T_2 = AC : BD. \qquad \text{Q.E.D.}$$

Otherwise thus: If the $\angle OAB = \theta$, we have

$$T_1 = 2P\cos\theta, \text{ and } T_2 = 2P\sin\theta;$$

$$\therefore T_1 : T_2 = \cos\theta : \sin\theta = OA : OB = AC : BD.$$

EXAMPLES. XXXVII. (Page 285.)

1. Let x and y be the stretched lengths of AB and BC, and T and T_1 be their tensions, respectively. Then for the equilibrium of W at B, we have $T = W + T_1$; and for the equilibrium of W at C, we have $T_1 = W$.

$$\therefore T = 2W;$$

$$\therefore 2W = \lambda\frac{x-c}{c} = 4W\frac{x-c}{c}.$$

$$\therefore x - c = \frac{c}{2}, \text{ and } x = \frac{3c}{2}.$$

Also
$$T_1 = W = 4W\frac{y-c}{c},$$

so that
$$y - c = \frac{c}{4}, \text{ and } y = \frac{5c}{4}.$$

2. Let ACB be the string, C be its middle point, and T and θ be (symmetrically) the tension and inclination to the vertical, respectively, of either AC or BC. Then we have

$$2T\cos\theta = W,$$

and
$$AB = (AC + CB)\sin\theta.$$

$$\therefore AC + CB - AB = \text{extension} = AB\frac{1-\sin\theta}{\sin\theta};$$

$$\therefore T = \lambda\frac{AB(1-\sin\theta)}{AB\sin\theta} = \frac{W}{\sqrt{3}}\cdot\frac{1-\sin\theta}{\sin\theta} = \frac{W}{2\cos\theta};$$

$$\therefore \cos\theta(1-\sin\theta) = \frac{\sqrt{3}}{2}\sin\theta.$$

$$\therefore\ 4(1-\sin^2\theta)(1-\sin\theta)^2=3\sin^2\theta.$$

$$\therefore\ 4-8\sin\theta-3\sin^2\theta+8\sin^3\theta-4\sin^4\theta=0.$$

$$\therefore\ 4(1-2\sin\theta)-3\sin^2\theta(1-2\sin\theta)+2\sin^3\theta(1-2\sin\theta)=0,$$

which is satisfied by

$$1-2\sin\theta=0,\qquad i.e.\ \sin\theta=\frac{1}{2},$$

$$i.e.\ \theta=30^\circ,\qquad i.e.\ 2\theta=60^\circ.$$

3. Here $AB=2a$, and the extension

$$=\frac{2a}{\sin\theta}-2c;$$

also $$2T\cos\theta=W,$$
as before;

$$\therefore\ T=\frac{W}{2\cos\theta}=\lambda\left(\frac{2a}{\sin\theta}-2c\right)\frac{1}{2c}.$$

$$\therefore\ \frac{W}{2\lambda}\tan\theta=\frac{a-c\sin\theta}{c},$$

i.e. $$\frac{W}{2\lambda}\tan\theta+\sin\theta=\frac{a}{c}.$$

4. Let W be the weight of the body, R be the reaction of the plane, and T be the tension.

We have

$$T=W\frac{x-a}{a}\ \ldots\ldots\ \ldots\ldots\ldots\ (1),$$

$$R=W\cos\alpha\ \ldots\ \ldots\ldots\ldots\ (2),$$

and $$T+R\tan\lambda=W\sin\alpha\ \ldots\ldots\ldots\ldots\ (3).$$

Substituting (1) and (2) in (3), we have

$$\frac{x-a}{a}+\cos\alpha\tan\lambda=\sin\alpha.$$

$$\therefore\ \frac{x}{a}=1+\sin\alpha-\cos\alpha\tan\lambda,$$

i.e. $$\frac{x}{a}=1+\sin(\alpha-\lambda)\sec\lambda.$$

5. Let AB, BC, CD and DA be the four rods, and A be the point of suspension. Let W be the weight of a rod, T be the tension of CA, and R be the action at C. Taking moments about A for the equilibrium of AB and BC as one system, we have

$$R\,.\,a\sqrt{2}=W\,.\,\frac{a}{2\sqrt{2}}+W\,.\,\frac{a}{2\sqrt{2}};$$

$$\therefore\ R=\frac{W}{2}.$$

Also, taking moments about B for the rod BC, we have

$$\frac{T}{2}\cdot\frac{a}{\sqrt{2}}=W\cdot\frac{a}{2\sqrt{2}}+R\cdot\frac{a}{\sqrt{2}},$$

whence $$T=2W.$$

But $$T=\lambda\frac{a\sqrt{2}-x}{x},$$

if x be the unstretched length;

$$\therefore\ 2W=W\frac{a\sqrt{2}-x}{x}.$$

$$\therefore\ 2x=a\sqrt{2}-x,$$

whence $$x=\frac{a\sqrt{2}}{3}.$$

6. 5 lbs. wt $=\lambda\frac{15-10}{10}$, so that $\lambda=10$ lbs. wt.

$\therefore$ tension, when the length is 12 inches, $=\lambda\frac{12-10}{10}=2$ lbs. wt.

$\therefore$ work done = extension × mean of the two tensions

$$=\frac{3}{12}\text{ ft.}\times\frac{5+2}{2}\text{ lbs. wt.}=\frac{7}{8}\text{ ft.-lb.}$$

7. When its extension is four inches, its tension is four pounds wt.

$\therefore$ required work done $=\frac{3}{12}$ ft. $\times\frac{1+4}{2}$ lbs. wt. $=\frac{5}{8}$ ft.-lb.

EXAMPLES. XXXVIII. (Pages 293—296.)

1. Scale—1 foot=1 inch. Draw $AB=6$ ins.; C is found by describing circles round A and B of radii 4 ins. and 5 ins. respectively. If D be the centre of BC, and G be the centre of gravity of the lamina,

$$GD=\frac{1}{3}AD.$$

Draw DO, perpendicular to BC, meeting the vertical (upwards) through G in O; join BO. On the scale 10 lbs. = 1 inch make OE in the vertical through G (downwards) = 30 lbs.; draw EF parallel to OB, meeting OD produced in F; then FO represents the pressure on the prop, and EF represents the strain at the hinge on the scale of 10 lbs. wt. to the inch. Also, the angle ABO must be measured with a protractor and is found to be $= 1° 40'$, nearly.

2. Draw BC horizontal and equal to 2 inches and CA vertical; with centre B and radius 6 inches draw an arc of a circle to meet CA in A. On the scale of 5 feet to 1 inch BA represents the position of the ladder.

Draw AD horizontal.

In the first case through the middle point G of BA draw GD vertical to meet AD in D and join BD. Then BD is the direction of the resultant force at B.

On DG take a point F where DF is 5 inches; on the scale of 30 lbs. wt. to an inch, DF therefore represents the weight. Draw FH horizontal to meet BD in H. Then HD represents the required reaction; if it be measured in inches, we shall have the reaction on the scale of 30 lbs. wt. to the inch.

In the second case let G_1 be the position of the weight so that G_1 bisects GA; the centre of gravity G' of the ladder and weight will therefore be such that

$$150GG' = 112G'G_1.$$

Hence
$$262GG' = 112GG_1 = 28AB,$$

$$\therefore \ GG' = \frac{14}{131} AB = \frac{84}{131} \text{ inches} = \cdot 641 \text{ inch},$$

on our scale.

Through G' draw $G'D'$ vertically to meet AD in D' and join BD'. Take $D'F'$ equal to $8\frac{11}{15}$, so that $D'F'$ represents 262 lbs. on the scale of 30 lbs. wt. to the inch. Draw $F'H'$ horizontally to meet $D'B$ in H'; then, as before, by measuring $H'D'$ we have the required reaction.

3. Draw the plane at an angle of 45° with the horizontal, and let P be the point of the mass at which the forces are applied. The resultant reaction is therefore opposite to the resultant of the two forces of 10 lbs. weight.

Measure equal distances PL and PM respectively up the plane and vertically downwards. Bisect LM in N and draw NP, producing it to Q. Then PQ is the direction of the resultant reaction and the tangent of the angle it makes with the normal to the plane is the required coefficient of friction. Draw PR perpendicular to the inclined plane. On PQ take Q such that PQ is 1 inch and draw QR perpendicular to PQ to meet PR in R. The number of inches in QR is then equal to the tangent of the angle RPQ, *i.e.* to the required coefficient of friction.

4. Take OP to represent the first force on the scale of 1 lb. $=1$ inch, and with centres O and P and radii to represent the other two forces on the same scale, describe circles meeting in Q; then OPQ is a triangle of forces, and if PR be drawn parallel to QO, and OP be produced to S, the required angles are RPQ, QPS and SPR, and they may now be measured by a protractor. Also, if A, B and C be the given points on the disc, by drawing through A a parallel to OP, through B a parallel to PR, and through C a parallel to PQ, the arrangement of the forces is determined by describing a circle on the segment AB having an angle in that segment equal to the angle SPR, and another on BC with an angle equal to the angle RPQ. These circles meet in a point H through which the three forces must act, so that their arrangement is along HA, HB and HC. These two circles are, of course, those circumscribing ABD and BCE, D and E being the points in which the lines through A and B meet, and the lines through B and C meet, respectively. The construction is, therefore, to bisect AD and BD and from their points of bisection erect perpendiculars to them, and the centre of the circle is the point in which the perpendiculars meet. Similarly with the other circle.

5. Draw the block on the scale given, making BC to CD as 3 to 5.

Let G be the centre of gravity of the block, and E and F the middle points of DC and CF respectively. Since EF is parallel to DB, the force we have to find acts in the direction EF. Since three of the forces go through E, the resultant reaction of the plane must also pass through E.

Draw GK horizontally in the direction ED and take GK equal to one-half of GE; then EK is the direction of the resultant reaction.

Take any vertical line LM, 4 inches in length, to represent the weight of 40 lbs.

Draw MN horizontally and parallel to EK, making MN equal to 1 inch, so that it represents the force 10 lbs. wt.

Draw NP parallel to EK and LP parallel to FE, and let them meet in P. Then, by the Polygon of Forces, PL represents the required force on the scale of 10 lbs. to the inch.

6. Draw a horizontal line AB equal to 3 inches and then BC vertically and equal to 1 inch, so that AC is the plane. Let E be the point at which the forces act. Draw EL perpendicular to the plane and equal to 2 inches, and through L draw LM perpendicular to EL (downwards) and equal to 1 inch. Then EM is the direction of the resultant reaction at E. Draw EH, making an angle of 40° with EC, so that it is the direction of the required force.

Draw EG vertically downwards and equal to 2 inches to represent the weight 100 lbs. of the body. Through G draw GF parallel to EH to meet ME produced in F.

Then GF represents the required force on the same scale that 2 inches represent 100 lbs.

For the second case produce ML to N so that LN equals ML. Then EN is now the direction of the resultant reaction at E. Draw EK making $\angle AEK$ equal to 40°. Through G draw GF' parallel to EK to meet NE produced in F'. On the same scale as before GF' now represents the required force.

7. Draw the triangle on a much smaller scale than that given.

Draw PQ parallel to AC and equal to 4 inches [scale—2 units of force = one inch] and QR, RS, ST parallel to BD, BA, and CB respectively and equal to 1, $\frac{3}{2}$ and 4 inches. Then PT represents the resultant in magnitude and direction.

Take any point O and join OP, OQ, OR, OS, OT.

On AC take any point a; draw ab parallel to OQ to meet BD in b, bc parallel to OR to meet BA in c, cd parallel to OS to meet BC in d, and de parallel to OT to meet in e a line drawn through a parallel to OP; then, by Art. 218, e is a point on the resultant.

Hence if through e we draw a straight line parallel to PT we have the line of action of the resultant. Also, on measuring PT, we have the magnitude on the scale one inch = 2 units of force.

8. (1) Draw LM, MN parallel to the forces 7, 5 and proportional to them. Take any pole O and join OL, OM, ON. On the first force take any point l; draw lm parallel to OM to meet the second force in m; through m and l draw mn, ln parallel to NO, OL to meet in n. By Art. 220 the resultant required passes through n. Draw nD parallel to the two forces to meet AB in D and measure AD.

(2) The construction is the same as in (1) except that now N is between L and M. D will be found to lie on DA produced.

9. Let the weights be hung on at points A, B, C. Draw a vertical line $LMNP$ making $LM = 2$ ins., $MN = 4$ ins., and $NP = 3$ ins. Take any convenient point O and join OL, OM, ON, OP. Take any point l on the vertical through A; draw lm parallel to OM to meet the vertical through B in m, and mn parallel to ON to meet the vertical through C in n. Through l, n draw lines lp, np parallel to OL and OP to meet in p. Then, by Art. 220, the vertical through p is the required line of action.

10. The construction in this case is the same as in the last except that there are four component forces so that we must start with a vertical line $LMNPQ$.

11. Draw the triangle ABC to scale. Draw KL vertically, and equal to 5 inches, to represent the weight on the scale of 10 lbs. to the inch.

Draw KM horizontal and LM parallel to CB. Then KM, LM represent the tension T_1 in AB and the thrust T_2 in CB.

Draw LN horizontal and MN vertical. Then LNM is a triangle of forces for the joint C, so that LN represents the reaction of the stop and NM the tension T_3 in CA.

On measuring KM, LM, NM in inches, and multiplying by 10, we have the values of T_1, T_2, T_3 in lbs. wt.

12. Let one inch represent one lb. The forces along AB, CD are equivalent to 3 lbs. parallel to AB through a point F on BC produced such that $CF=\frac{1}{3}BC$. The forces along BC and AD are equivalent to 2 lbs. parallel to AD through a point E on BA produced such that $AE=BA$.

Through F, E draw lines parallel to BA and AD and let them meet in N. On FN produced and NE, produced if necessary, take distances NK and NL equal to 3 and 2 inches respectively. Complete the parallelogram $KNLM$. Then NM is the required resultant and will be found to be 3·6 inches. If MN produced meet BC in G, then clearly angle $BGN=\angle LNM=\tan^{-1}\frac{3}{2}$, and FG will be found to be $\frac{5}{3}BC$.

13. Let D be the middle point of AB and F the point of the ground to which the rope is attached. Draw DE vertically to meet CF in E.

By symmetry the resultant action of the two beams is along DC. The triangle DCE is therefore a triangle of forces for this resultant action, the tension of the rope and the weight 10 cwt.

Since $\angle EDC=90°-70°=20°$, and $\angle DCF=70°-50°=20°$, we have EC equal to ED, so that the tension of the rope is 10 cwt.

Hence the action along DC balances two forces each equal to 10 cwt. inclined to it at 20° each.

Consider now the triangle ACB where the angles ACD and BCD are each 20°. It follows that the action along each of the beams BC and AC is 10 cwt.

14. Let 1 ft. be represented by half an inch. Draw AE ($=7\frac{1}{2}$ ins.) horizontally and then EC ($=10$ inches) vertically. On AE take AF ($=6$ ins.) and FB vertically ($=4\frac{1}{2}$ inches). Then AB is $7\frac{1}{2}$ inches, so that it represents the beam whose length is 15 ft.

Bisect AB in G. Draw GL vertically to meet the line CB produced in L and join AL. Draw BM parallel to LA to meet LG produced in M. Then LBM is a triangle of forces for the system. Measure the distances ML, LB, and BM.

The $$\frac{\text{tension}}{LB}=\frac{140 \text{ lbs. wt.}}{ML}=\frac{\text{total pressure}}{BM}.$$

Again, measure the distance DL, where GL meets AF in D.

The coefficient of friction then

$$=\tan ALD=\frac{AD}{DL}=\frac{3}{DL},$$

and is therefore known.

15. Draw a triangle ABC whose sides BC, CA, and AB are proportional to 9, 8 and 7 [equal, say, to $4\frac{1}{2}$, 4 and $3\frac{1}{2}$ inches].

Let D, E, F be the middle points of the sides BC, CA, and AB.

Let T_1, T_2, and T_3, be the actions in the sides BC, CA, and AB.

Produce AD and on it take a line MN, $2\frac{1}{2}$ ins. in length, to represent P (=50 lbs. wt.), so that the scale is 20 lbs. wt. to an inch.

Through N draw NL parallel to EB and through M draw ML parallel to CF. Then MNL is a triangle of forces for P, Q and R. Hence, by measuring NL and LM, the forces Q and R are known by allowing 20 lbs. wt. for each inch.

Again, draw NO and MO parallel respectively to AB and CA. It will then be found that LO is parallel to BC. The triangle MNO is therefore a triangle of forces for the forces at the point A, so that T_2 and T_3 are found by measuring the lengths MO and NO.

Similarly LOM is a triangle of forces for the point C, so that T_1 is known by measuring OL.

By drawing the triangle MNL and constructing the point O as in the last example we obtain the forces proportional to MN, NL, and LM, and the stresses in the bars proportional to OL, OM, and ON.

16. Draw the normals at A and B, the first being below the rod and the second above. Through A and B draw, on the sides toward the upper end of the rods, two lines inclined at λ, the angle of friction, to these perpendiculars. Through the point where these two meet draw the vertical to meet the rod in G. This point will be the limiting position of the centre of gravity. If the inclination of the rod to the horizon be $<\lambda$, then the inclination of the normal at B to the vertical will be less than λ, and then the line drawn through B will meet the line drawn through A toward the *lower* end of the rod. In this case the *limiting* position of G would appear to be between A and B, which is impossible, *i.e.* there is no limiting position, *i.e.* if the inclination of the rod to the horizon be $<\lambda$, the rod will rest in any position in which the centre of gravity is above B.

17. Replace the wt. of 100 lbs. by two weights, each of 50 lbs., placed at A and B.

Let 25 lbs. wt. be represented by one inch. Draw AL vertical and equal to two inches. Through L draw LM, parallel to AB, to meet CA produced backward in M. Then LM represents the action along AB. Draw MN perpendicular to AL. Then MN represents the force P and LN represents the tension in BD – 50 lbs. wt. On measurement, MN is one inch, so that P=25 lbs. wt.

18. Let R, S be the reactions at B and C. Let W be represented by two inches. Draw $B'C'$ vertical and $=2$ inches. Take any point O. Join OB', OC'. On the line of action of R take any point α; draw $\alpha\beta$ parallel to OB' to meet vertical through E in β; draw $\beta\gamma$ parallel to OC' to meet the vertical through C in γ; join $\alpha\gamma$ and draw OD' parallel to $\alpha\gamma$ to meet $B'C'$ in D'. Then $C'D'$ represents S and $D'B'$ represents R.

Let DF meet the line of action of R in G and join GA; then GA is the line of action of the action at the hinge A.

Draw $D'E'$ parallel to FD and $E'B'$ parallel to GA. Then $B'C'D'E'$ is the polygon of forces [reducing to a triangle of forces] for the rod AC. Thus $D'E'$ represents the tension of the string. On measurement $D'E'=$one inch; $\therefore$ the tension$=\frac{W}{2}$.

19. Let P, Q, R be the required forces along AC, AF, and DE respectively. The two forces at E have a resultant passing through E and the two forces at A have one passing through A. If therefore we are to have equilibrium the lines of action of the two resultants must be EA and AE. Let 1 inch represent 10 lbs. wt. Draw LM parallel to EC and 4 inches in length. Draw MN parallel to AE and NL parallel to DE. Then NL represents R. Draw MO parallel to AC and NO parallel to FA. Then MO, ON, NM represent P, Q, and the resultant at E and therefore give equilibrium. On measurement, $MO=1$ inch, $ON=1{\cdot}732$ and $NL=3{\cdot}464$ inches, $\therefore$ etc.

20. By symmetry the value of R is 10 cwt.

Draw KL vertically upwards and of length 2 inches

[scale—5 cwt. $=1$ inch].

Draw LM horizontal and KM parallel to the left-hand side rod. Then LM, KM represent T_4 and T_1.

At M draw MN vertically and equal to 1 inch, and through N draw NP parallel to MK, and through K draw KP parallel to the right-hand side rod. Then NP, PK represent T_2 and T_3. Produce KP to meet MN in Q. Then PNQ is the triangle of forces for the upper joint, so that NQ represents 10 cwt. $+T_5$. But, by geometry, we easily have $NQ=3MN=15$ cwt.

Therefore $T_5=5$ cwt. (T_5 acting downwards at the vertex of the system).

On measurement, we have the results in the answer.

T_4 and T_5 are ties; the others are struts.

21. The reaction R at A is clearly 5 cwt. Draw KL vertical upwards and equal to 2 inches [scale—one inch$=2\frac{1}{2}$ cwt.]; draw KM parallel to CA and LM parallel to DA. Then MK, ML represent T_1 and T_2.

Draw MP parallel to DB to meet LK produced at P. Then LP represents the total vertically downward thrust at D.

Draw MO horizontal to meet KP in O. Then

$$LP = 2LO = 2LK + 2KO = 10 \text{ cwt.} + 2KO.$$

Therefore T_3 is represented by $2KO$.

Measure therefore MK, ML, and $2KO$; on the scale 1 inch $= 2\frac{1}{2}$ cwt. we have T_1, T_2, T_3.

T_1 is a strut; T_2 and T_3 are ties.

22. Let T_1, T_2, T_3, T_4 be the actions in the parts AB, AC, CB, CD.

Draw KL vertically downwards and equal 3 inches
[scale—5 cwt. $=$ 1 inch].

Draw KM, LM parallel to CA, BA. Then KLM is a triangle of forces for the point A and LM, KM represent T_1 and T_2 respectively.

Draw MN vertically to meet KN, drawn through K parallel to DC, in N. Then KMN is a triangle of forces for the joint C, and hence MN, NK represent T_3 and T_4.

On measuring LM, KM, KN we have T_1, T_2, T_4 on the scale 1 inch $=$ 5 cwt.

T_1 is a strut; T_2 and T_4 are ties.

23. This example is drawn just like the one given on page 292; in this case the point δ will come below β instead of above.

24. Construct the figure $ABCD$ of the shape given. Draw KL equal to 4 inches to represent the action at A [it must act along CA for equilibrium of the whole figure]; the scale is one inch $=$ 10 lbs.

Draw LM, KM parallel to AB, AD; then LM, MK represent T_1 and T_4 where T_1, T_2, T_3, T_4, T_5 are the actions in the rods AB, BC, CD, DA, DB respectively.

Draw LN, MN parallel to DB, BC respectively; then they represent T_5 and T_2; through M, N draw MP and NP parallel to CD and AC; then MP will represent T_3 and NP will be found to be 4 inches as it should.

On measurement we have the actions on the scale 10 lbs. $=$ 1 inch.

T_5 is a strut; the others are ties.

EXAMPLES. XXXIX. (Pages 316, 317.)

1. Let $ABCD$ be the rhombus, A being the highest point. Let E, F, G, H be the middle points of AB, BC, CD, DA. Then the depths of F and G below A are three times those of E or H, so that, if in any displacement E ascends x, then the total work done against the weights

$$=W.x+W.x+W.3x+W.3x=8Wx.$$

Let now the displacement be such that α becomes $\alpha+\theta$, so that E goes to L. Draw LN vertical and EN horizontal. Then $NL=x$ and total work against the weights $=8W.NL$. Also work done by rod $=T\times 2EN$, where T is its tension, since the displacement is supposed to be symmetrical, and so H is displaced similarly to E.

$$\therefore\ T\times 2EN=8W.NL.$$

$$\therefore\ T=4W.\frac{NL}{EN}=4W\ \tan LEN=4W\cot AEH=4W\tan\alpha,$$

since θ is supposed to be infinitely small, and thus the angle AEL is indefinitely near a right angle.

2. Let $ABCD$ be the rhombus having AD fixed in a horizontal position; let AC be the shorter diagonal, so that ADC is an acute angle. Let the diagonals intersect in O; from E, the middle point of CD, draw EF perpendicular to AD. Let $EF=x$.

Let $$\angle ADB=\angle CDB=a.$$
Then $$a=2\,.\,AO=2b\sin a,$$
and $$x=\tfrac{1}{2}b\sin 2a.$$

Let a displacement take place by which a becomes $a+\theta$, a becomes $a+c$, and x becomes $x+y$, where θ, c, and y are small.

Then $$a+c=2b\sin(a+\theta)\quad\text{and}\quad x+y=\tfrac{1}{2}b\sin(2a+2\theta).$$

$$\therefore\ c=2b[\sin\overline{a+\theta}-\sin a]=4b\cos\left(a+\frac{\theta}{2}\right)\sin\frac{\theta}{2},$$

and $$y=\frac{b}{2}[\sin(2a+2\theta)-\sin 2a]=b\cos(2a+\theta)\sin\theta.$$

Since depth of the middle point of BC is twice that of E, the virtual work of the weights of the rods

$$=W\,.\,y+W\,.\,y+W\,.\,2y=4W\,.\,y.$$

The equation of virtual work gives

$$T\,.\,c-4W\,.\,y=0.$$

$$\therefore\ T=4W\times\frac{y}{c}=4W\times\frac{\cos(2a+\theta)\sin\theta}{4\cos\left(a+\frac{\theta}{2}\right)\sin\frac{\theta}{2}}$$

$$=2W\frac{\cos(2a+\theta)\cos\frac{\theta}{2}}{\cos\left(a+\frac{\theta}{2}\right)}=2W\frac{\cos 2a}{\cos a},\ \text{when }\theta\text{ is zero,}$$

$$=2W\frac{1-2\sin^2 a}{\sqrt{1-\sin^2 a}}=2W\frac{4b^2-2a^2}{2b\sqrt{4b^2-a^2}}=2W\frac{2b^2-a^2}{b\sqrt{4b^2-a^2}}.$$

3. Produce AB to K so that $\angle KBC=60°$. Let a be length of a side of the hexagon; then

$$FC=2+2a\cos 60°=2a.$$

Let KBC become $60°+\theta$, where θ is small, and FC become $2a+x$. Then

$$2a+x=a+2a\cos(60°+\theta)=a+2a[\cos 60°\cos\theta+\sin 60°\sin\theta];$$

$$\therefore\ x=a[\cos\theta-1+\sqrt{3}\sin\theta].$$

Let y be the height of the centre of gravity of BC above its original position. Then

$$y=\frac{a}{2}\sin(60°+\theta)-\frac{a}{2}\sin 60°$$

$$=\frac{a}{2}\,.\,2\cos\left(60°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}=a\cos\left(60°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

The centres of gravity of CD and EF rise a distance $3y$, and that of DE rises $4y$. The equation of virtual work gives

$$T\times x=W.y+W.3y+W.4y+W.3y+W.y=12W.y;$$

$$\therefore\ T=12W\frac{\cos\left(60^\circ+\frac{\theta}{2}\right)\sin\frac{\theta}{2}}{\sqrt{3}\sin\theta-2\sin^2\frac{\theta}{2}}=12W\frac{\cos\left(60^\circ+\frac{\theta}{2}\right)}{2\sqrt{3}\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}}$$

$$=12W\frac{\cos 60^\circ}{2\sqrt{3}},\text{ when }\theta\text{ becomes zero,}$$

$$=\frac{6W}{2\sqrt{3}}=W\sqrt{3}.$$

4. Let OA, OB, OC be the three rods, O being the upper end of each.

Let the vertical through O meet the plane ABC in O', and let $OO'=h$. The sides of the triangle ABC are each $2b$, and O' is the circumcentre of the triangle, so that

$$O'B=\frac{BC}{2\sin 60^\circ}=\frac{2b}{\sqrt{3}}.$$

$$\therefore\ h^2=OB^2-O'B^2=a^2-\frac{4b^2}{3}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Let the vertex O be displaced through a distance y, so that h becomes $h-y$, and b becomes $b+x$. Then (1) gives

$$(h-y)^2=a^2-\frac{4}{3}(b+x)^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Subtracting (2) from (1), we have

$$2hy-y^2=\frac{4}{3}(2bx+x^2).$$

Neglecting the squares of small quantities, this equation gives

$$2hy=\frac{8}{3}bx,\quad i.e.\quad y=\frac{4}{3}\frac{b}{h}x\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

If T be the required tension, the equation of virtual work gives

$$3\times T\times x=W\times y+3w\times\frac{y}{2}.$$

$$\therefore\ T=\frac{2W+3w}{2}\frac{y}{3x}=\frac{2W+3w}{2}\times\frac{4}{9}\frac{b}{h}=\frac{2}{3}(2W+3w)\frac{b}{\sqrt{9a^2-12b^2}},$$

by equation (1).

5. Let $ABCD$ be the square, A being the highest point, so that

$$\angle CAD=\angle CAB=45^\circ.$$

Let a displacement be considered by which $\angle CAD$ becomes $45°+\theta$, where θ is small. Then change in the length BD

$$=2a\sin(45°+\theta)-2a\sin 45°$$

$$=2a\times 2\cos\left(45°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}=4a\cos\left(45°+\frac{\theta}{2}\right)\sin\frac{\theta}{2},$$

where a is the side of the square.

The C.G. of AD rises through a distance

$$=\frac{a}{2}\cos 45°-\frac{a}{2}\cos(45°+\theta)=a\sin\left(45°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

The C.G. of BC or CD rises through three times this distance. Also B and D rise twice and C four times this distance. The equation of virtual work gives

$$T\times 4a\cos\left(45°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}$$

$$=(W+3W+3W+W+2W+4W+2W)\times a\sin\left(45°+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

$$\therefore\ T=\frac{16Wa}{4a}\times\frac{\sin\left(45°+\frac{\theta}{2}\right)}{\cos\left(45°+\frac{\theta}{2}\right)}=4W,\ \text{when } \theta \text{ is zero.}$$

6. Let the angle BAC be α, and let a displacement be imagined by which α becomes $\alpha+\theta$. The increase in the length of DB

$$=2a\sin(\alpha+\theta)-2a\sin\alpha=4a\cos\left(\alpha+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

The decrease in the height above A of the middle point of AB

$$=\frac{a}{2}\cos\alpha-\frac{a}{2}\cos(\alpha+\theta)=a\sin\left(\alpha+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

The height of the middle points of BC or CD decreases by three times this quantity. The equation of virtual work gives

$$T\times 4a\cos\left(\alpha+\frac{\theta}{2}\right)\sin\frac{\theta}{2}=(W+3W+3W+W)\times a\sin\left(\alpha+\frac{\theta}{2}\right)\sin\frac{\theta}{2},$$

i.e. $$T=\frac{8Wa}{4a}\tan\left(\alpha+\frac{\theta}{2}\right)=2W\tan\alpha,$$

when θ is made zero,

$$=2W\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}=2W\frac{\frac{l}{2a}}{\sqrt{1-\frac{l^2}{4a^2}}}=2W\frac{l}{\sqrt{4a^2-l^2}}.$$

7. In the position of equilibrium let y be the radius of the string, and x its depth, so that

$$y = x \tan a \quad \text{............ ....} \quad (1).$$

Let a displacement be imagined in which x becomes $x+h$ and y becomes $y+k$ so that

$$y+k=(x+h)\tan a \quad \text{.................... ....} \quad (2).$$

Subtracting (1) from (2) we have

$$k = h\tan a.$$

The equation of Virtual Work gives

$$T[2\pi(y+h)-2\pi y]=W.h,$$

i.e.
$$T=\frac{W}{2\pi}\frac{h}{k}=\frac{W}{2\pi}\cot a.$$

But, by Hooke's Law,

$$T=\lambda\frac{2\pi y-2\pi a}{2\pi a}.$$

$$\therefore y=a\left[1+\frac{T}{\lambda}\right]=a\left[1+\frac{W}{2\pi\lambda}\cot a\right].$$

8. Let the rod AB touch the circle in D; let G be its centre of gravity and O the centre of the circle.

The height of G above O = ht. of A above O − ht. of A above G

$$=OD\operatorname{cosec}\theta-b\cos\theta=\frac{a}{\sin\theta}-b\cos\theta.$$

Let the displacement be such that θ becomes $\theta+a$, where a is small. The reaction at D does not come into the equation of virtual work, since the displacement of D is perpendicular to the direction of the reaction; nor does the mutual reaction at A between the rods. Hence the equation of virtual work gives only

$$W\left[\left(\frac{a}{\sin\theta}-b\cos\theta\right)-\left(\frac{a}{\sin\overline{\theta+a}}-b\cos\overline{\theta+a}\right)\right]=0.$$

Now, since a is very small,

$$\sin(\theta+a)=\sin\theta+\cos\theta.a,$$

and
$$\cos(\theta+a)=\cos\theta-\sin\theta.a,$$

so that
$$\frac{1}{\sin(\theta+a)}=\frac{1}{\sin\theta(1+a\cot\theta)}$$

$$=\frac{1}{\sin\theta}(1-a\cot\theta).$$

Hence, substituting in the above and cancelling like terms, we have

$$W\left[\frac{aa}{\sin\theta}\cot\theta-ba\sin\theta\right]=0,$$

i.e. $$\frac{a}{\sin\theta}\cot\theta = b\sin\theta,$$

i.e. $$a\cos\theta = b\sin^3\theta.$$

9. Let $ABCD$ be the rhombus; let AC, BD meet in O and put

$$AO = OC = x, \quad BO = OD = y.$$

Let T, T' be the tensions of AC and BD. Let the displacement be such that x and y become $x+h$ and $y+k$. Since diagonals of a rhombus intersect at right angles, we have

$$x^2 + y^2 = AB^2 = (x+h)^2 + (y+k)^2,$$

so that $$0 = 2xh + h^2 + 2yk + k^2,$$

i.e. $0 = xh + yk$, on neglecting squares of small quantities.

$$\therefore \frac{h}{k} = -\frac{y}{x}.$$

But the equation of virtual work gives

$$T \times \text{change in length of } AC + T' \times \text{change of } BD = 0,$$

i.e. $$T \times 2h + T' \times 2k = 0.$$

$$\therefore \frac{T}{T'} = -\frac{k}{h} = \frac{x}{y}.$$

$$\therefore T : T' :: x : y :: AC : BD.$$

EASY MISCELLANEOUS EXAMPLES. (Pages 318–320.)

1. If $$\sin\alpha = \frac{12}{13}, \text{ then } \cos\alpha = \frac{5}{13}.$$

Hence, by Art. 27, we have

$$R = \sqrt{(13)^2 + (14)^2 + 2 \,.\, 13 \,.\, 14\left(-\frac{5}{13}\right)} = 15 \text{ lbs. wt.}$$

Also in the second figure of Art. 27, we have

$$OA = BC = 14,\ AC = OB = 13,\ CD = 12, \text{ and } DA = 5;$$

$$\therefore \tan COD = \frac{CD}{OD} = \frac{12}{14-5} = \frac{12}{9} = \frac{4}{3},$$

i.e. $$\angle COD = \tan^{-1}\frac{4}{3}.$$

2. Proceed as in Examples II. 4.

3. Proceed as in Examples V. 7.

4. Proceed as in Examples IV. 1, with 100 lbs. wt. for 80. The resultant

$$= OC = OB\cos 30^\circ = 50\sqrt{3} = 86{\cdot}6\ldots \text{ lbs. wt.}$$

5. The forces represented by AG and CG have resultant represented by $2EG$; and the forces represented by BG and DG have resultant represented by $2FG$; but EG and FG are equal and opposite; hence the four forces are in equilibrium.

6. If the pole meet the wall at the point A, the moments about A in the two cases, which measure the tendency of the pole to break at A, must be the same. Hence, if W be the weight of the pole, and G be its centre of gravity, we have

$$W . AG + 28 \times 12 = W . AG + 8 \times 14 \times x,$$

where x is the required distance measured in feet;

$$x = \frac{28 \times 12}{8 \times 14} = 3 \text{ feet.}$$

7. Let AB be the rod, C be its middle point where its weight, 4 oz., acts, D be the edge of the table, and W oz. be the required weight attached to B. Then $DB = 1$ ft., and $DC = \frac{1}{2}$ ft. Taking moments about D, we have

$$W . 1 = 4 . \frac{1}{2},$$

whence $W = 2$ oz.

8. Let AB be the beam, A being the lower end, G be its middle point, W be the weight attached to the string, and P and Q be the vertical and horizontal components respectively of the pressure (R) on A. Then the tension of the string is equal to W. Resolving vertically and horizontally, we have

$$P = 30, \text{ and } Q = W.$$

Also, taking moments about A, we have

$$W . AB \cos 60° = 30 . \frac{AB}{2} . \sin 60°;$$

$$\therefore\ W = 15 \tan 60° = 15\sqrt{3} = 26 \text{ lbs. wt., nearly,}$$

and $\quad R = \sqrt{P^2 + Q^2} = \sqrt{900 + 675} = \sqrt{1575} = 40$ lbs. wt., nearly.

Otherwise thus: The directions of the three forces R, W the tension of the string, and the weight of the beam, 30 lbs., meet in a point O, and G is vertically below O. Let OG meet the ground in the point D. Then

$$OD = \frac{1}{2} AB = AG; \quad AD = AG \sin 60° = \frac{AG\sqrt{3}}{2};$$

and

$$AO = AG\sqrt{1 + \frac{3}{4}} = \frac{AG\sqrt{7}}{2}.$$

Resolving vertically, we have

$$R.\cos AOG=30, \text{ i.e. } R.\frac{2}{\sqrt{7}}=30.$$

$$\therefore\ R=15\sqrt{7}=\sqrt{1575}=40 \text{ lbs. wt., nearly};$$

also, we have

$$W=\sqrt{R^2-(30)^2}=\sqrt{675}=26 \text{ lbs. wt., nearly.}$$

9. Proceed as in Examples XVII. 20.

10. Let G be the centre of gravity, and draw GM perpendicular to BA. Then we have

$$BM=\frac{4.8+5.0+6.0}{4+5+6}=\frac{32}{15};$$

$$GM=\frac{4.0+5.0+6.11}{4+5+6}=\frac{66}{15}=3\times\frac{22}{15};$$

and $$AM=8-\frac{32}{15}=\frac{88}{15}=4\times\frac{22}{15};$$

hence $$AG=\sqrt{AM^2+GM^2}=\frac{22}{15}\sqrt{4^2+3^2}=\frac{22}{3}=7\tfrac{1}{3} \text{ ins.}$$

11. Proceed as in Examples XVIII. 4.

12. The centre of gravity of the complete hexagon being at O, G the required centre of gravity is evidently on the line joining O to the middle point L of the side opposite to that cut away, by symmetry. Let H be the centre of gravity of the triangle cut away, and K be the middle point of its base. Then we have

$$OG : OH=\text{wt. at } H: \text{wt. at } G=\frac{W}{6}:\frac{5W}{6}=1:5;$$

$$\therefore\ OG=\frac{1}{5}OH=\frac{1}{5}\cdot\frac{2}{3}\cdot OK=\frac{2}{15}OK=\frac{2}{15}OL;$$

$$\therefore\ OG : GL=2:13.$$

13. Suppose each penny to project an equal distance (a, say) beyond the next one beneath it. Then the centre of gravity of the upper five pennies will be at the centre of the middle one, and, for equilibrium, must be vertically above the edge of the lowest penny. Hence, if r be the radius of a penny, we have $3a=r$; and the total projection of the top penny $=5a=\frac{5}{3}r=\frac{5}{6}$ diameter.

14. Let P lbs. and Q lbs. be the required weights, at distances from the fulcrum in the ratio 3 : 2 respectively. Then we have

$$3P=2Q;$$

but $$P+Q=20;$$

$$\therefore\ 3P=2(20-P), \text{ i.e. } 5P=40, \text{ and } P=8 \text{ lbs.}$$

$$\therefore\ Q=12 \text{ lbs.}$$

15. If P lbs. wt. be the force, acting vertically upwards, at the other end, then, taking moments about the fulcrum, we have

$$P\times 5=6\times 3+10\times\frac{5}{2}+3\times 1=46.$$

$$\therefore\ P=9\tfrac{1}{5}\text{ lbs. wt.}$$

Hence the pressure on the fulcrum

$$=3+10+6-P=19-9\tfrac{1}{5}=9\tfrac{4}{5}\text{ lbs. wt.}$$

16. Taking the figure in Art. 147, with 5 pulleys, we have

$$T_5=P,\ T_4=2T_5-P=P,$$

$$T_3=2T_4-P=P,\ T_2=2T_3-P=P,\ T_1=2T_2-P=P,$$

and $$W=2T_1-P=P.$$

17. Taking the figure in Art. 150, with 5 pulleys, we have

$$T_1=P,\ T_2=2T_1=2P,\ T_3=2T_2=4P,\ T_4=2T_3=8P,$$

and $$T_5=2T_4=16P.$$

Hence [cf. Art. 152] D and G being the points of attachment of the two extreme strings, we have

$$DX=\frac{T_5\times 0+T_4\times 1+T_3\times 2+T_2\times 3+T_1\times 4}{T_5+T_4+T_3+T_2+T_1}=\frac{26}{31}\text{ in.};$$

$$\therefore\ GX=4-\frac{26}{31}=\frac{98}{31}\text{ in.};$$

$$\therefore\ DX:XG=26:98=13:49.$$

18. Proceed as in Examples XXVI. 8.

19. If θ be the angle the beam makes with the horizon, moments about the fulcrum give

$$W'.k\sin\theta=S(a\cos\theta-h\sin\theta);$$

whence $$\tan\theta=\frac{Sa}{W'k+Sh},\ \textit{i.e.}\ \theta=\tan^{-1}\frac{Sa}{W'k+Sh}.$$

20. With the notation of Art. 171, we have here

$$GC=2\text{ ins.},\ P=4\text{ oz., and }W'=2\text{ lbs.}$$

Hence, for O, we have

$$2\times 16\times 2=4.CO.$$

$$\therefore\ CO=16\text{ ins.}$$

$$\therefore\ GO=18\text{ ins.}=1\tfrac{1}{2}\text{ ft.}$$

Also, the graduations are given by

$$OX=\frac{W}{P}.CA,$$

and are in Arith. Prog. with the common difference $CA=4$ ins., P, or 4 oz., being the amount indicated by 1 in the scale of graduation.

21. By Art. 178, the mechanical advantage

$$=\frac{20}{\cdot 75}=20\times\frac{4}{3}=\frac{80}{3}=26\tfrac{2}{3}.$$

22. Let a be the inclination of the plane to the horizon, W be the weight of the body, and R be the normal reaction. Then resolving perpendicular and parallel to the plane, we have

$$R=\frac{W}{2}\sin a+W\cos a,$$

and
$$\frac{W}{2}\cos a+\mu R=W\sin a,$$

$$\therefore\ \frac{W}{2}\cos a+\frac{\mu W}{2}\sin a+\mu W\cos a=W\sin a;$$

dividing by $\frac{W}{2}\cos a$, we have

$$1+\mu\tan a+2\mu=2\tan a.$$

$$\therefore\ \mu\left(2+\frac{3}{4}\right)=\frac{3}{2}-1=\frac{1}{2},\ \text{whence}\ \mu=\frac{2}{11}.$$

23. Take the figure on p. 260, but the wall smooth, *i.e.* no friction at B. We have

$$\cos\theta=\frac{6}{30}=\frac{1}{5},$$

and
$$\therefore\ \sin\theta=\sqrt{1-\frac{1}{25}}=\frac{2\sqrt{6}}{5}.$$

Let the man be on the *point* of slipping when he has ascended a distance $AX\ (=x)$. Resolving vertically and horizontally, we have $R=5W$, and

$$S=\mu R=\frac{1}{2}\times 5W=\frac{5W}{2}.$$

Also, taking moments about A, we have

$$W\,.\,15\cos\theta+4W\,.\,x\cos\theta=S\,.\,30\sin\theta.$$

$$\therefore\ 15W+4Wx=\frac{5W}{2}\,.\,60\sqrt{6},\ \text{whence}\ x=\frac{150\sqrt{6}-15}{4},$$

which is >30; hence the man can ascend the whole length of the ladder without its being on the point of slipping.

24. Volume of the chalk

$$=\pi\times 5^2\times 600\ \text{cub. ft.};$$

weight of the chalk

$$=25\times\pi\times 600\times 2\cdot 3\times 62\tfrac{1}{2}\ \text{lbs.};$$

depth of the centre of gravity

$$=300\ \text{ft.};$$

work to be done

$$=25\times\pi\times 600\times 2{\cdot}3\times\frac{125}{2}\times 300 \text{ ft.-lbs.}$$

in $(12\times 8\times 60)$ minutes; hence the required H.P.

$$=\frac{25\times\frac{22}{7}\times 600\times\frac{23}{10}\times\frac{125}{2}\times 300}{12\times 8\times 60\times 33000}=10\tfrac{935}{1344}.$$

HARDER MISCELLANEOUS EXAMPLES.
(Pages 320—332.)

1. If a, b and c be the sides of the triangle ABC, since OA, OB and OC are all equal, the forces may be represented by $a.OA$, $b.OB$ and $c.OC$; and the resultant of $b.OB$ and $c.OC=(b+c)\,OE$, along OE, where E is the point in BC such that

$$b.BE=c.CE,$$

$$i.e.\ BE:CE=BA:CA,$$

and thus AE bisects the angle A. Hence the resultant of all three forces

$$=(b+c+a)\,OI,$$

where I is the point in AE such that

$$(b+c)\,IE=a.IA,$$

$$i.e.\ IA:IE=b+c:a.$$

$$\therefore\ AE:IE=a+b+c:a;$$

thus the distance of I from $BC=\frac{S}{s}=$ the radius of the inscribed circle; but the only point on the bisector of A at this distance from BC is the centre of the inscribed circle; hence I must be this centre.

2. Let P, Q and S be the forces acting along the sides BC, CA and AB respectively. Let O' be the orthocentre, and G be the centre of gravity of the triangle; also let D, E and F be the feet of the perpendiculars from O' on BC, CA and AB respectively, and GH be perpendicular to BC. Then

$O'D=2R\cos B\cos C$, $O'E=2R\cos C\cos A$, and $O'F=2R\cos A\cos B$, where R is the radius of the circumscribing circle; also

$$GH=\frac{1}{3}AD=\frac{1}{3}\cdot\frac{2\Delta}{BC}.$$

Since the resultant of P, Q and S passes through O' and G, the algebraical sum of the moments of P, Q and S about each of those points is zero. Hence we have

$$P.2R\cos B\cos C+Q.2R\cos C\cos A+S.2R\cos A\cos B=0,$$

and $$P\cdot\frac{2\Delta}{3a}+Q\cdot\frac{2\Delta}{3b}+S\cdot\frac{2\Delta}{3c}=0;$$

whence $\quad P\cos B\cos C+Q\cos C\cos A+S\cos A\cos B=0,$

and $\quad P\sin B\sin C+Q\sin C\sin A+S\sin A\sin B=0.$

Hence, solving by cross multiplication, we have

$$\frac{P}{\sin 2A\sin(B-C)}=\frac{Q}{\sin 2B\sin(C-A)}=\frac{S}{\sin 2C\sin(A-B)}.$$

Again, if O and I be the centres of the circumscribing and inscribed circles, and R and r be the radii, respectively, the perpendiculars from O on the sides BC, CA and AB are $R\cos A$, $R\cos B$ and $R\cos C$ respectively; and the perpendiculars from I on the sides are each equal to r. If the resultant pass through O and I, the algebraical sum of the moments of P, Q and S about each of those points is zero. Hence we have

$$P\cdot R\cos A+Q\cdot R\cos B+S\cdot R\cos C=0,$$

and $$P\cdot r+Q\cdot r+S\cdot r=0.$$

Hence, solving by cross multiplication, we have

$$\frac{P}{\cos B-\cos C}=\frac{Q}{\cos C-\cos A}=\frac{S}{\cos A-\cos B}.$$

3. As in Ex. 11, p. 146, the resultant of the first three forces is $3PG$ acting toward G the centre of gravity of the triangle ABC. So the resultant of the second three forces is $3GQ$. Hence the resultant of all the forces is, by the triangle of forces, a force through G parallel to PQ and equal to $3PQ$.

4. As in Ex. 11, p. 146, the resultant of forces AT, BT, and CT is $3GT$ where G is the centre of gravity of the triangle ABC. But it is well known that O, G, and T lie on a line and that $3GT=2OT$.

Hence the required resultant is represented by $2OT$.

5. Let ABC be the triangle, and I_1, I_2 and I_3 be the centres of the escribed circles. Then I_2I_3 bisects the external angle at A, and therefore the particles at I_2 and I_3 are as

$$\frac{1}{AI_2\cos\frac{A}{2}}:\frac{1}{AI_3\cos\frac{A}{2}}=AI_3:AI_2.$$

Hence their centre of gravity is at A, and the centre of gravity of the three particles lies on AI_1 which bisects the angle A.

Similarly the centre of gravity lies on the bisector of the angle B. Hence it is at the centre of the inscribed circle.

6. When suspended from P, the centre of gravity of the trapezoid $ABCP$ must be in the vertical through P, and therefore in the parallel PQ to AB through P. Let $AP=x$, $AD=a=BC$, and $AB=b$. Bisect

BC in F, and AP in E; join E and F. The centre of gravity of the trapezoid is at G, where

$$EG : GF = 2a + x : a + 2x,$$

and G is vertically below P; $\therefore$ PQ is parallel to AB;

$$\therefore\ EP : QF = EG : GF,$$

i.e.
$$\frac{x}{2} : \frac{a}{2} - x = 2a + x : a + 2x.$$

$$\therefore\ \frac{ax}{2} + x^2 = a^2 - 2ax + \frac{ax}{2} - x^2.$$

$$\therefore\ 2x^2 + 2ax - a^2 = 0.$$

$$\therefore\ 3x^2 = a^2 - 2ax + x^2 = (a - x)^2.$$

$$\therefore\ x : a - x = 1 : \sqrt{3},$$

$$\text{i.e. } AP : PD = 1 : \sqrt{3}.$$

7. Let D be the middle point of AC, BF be the perpendicular on AC produced, and X be the least weight required. Let the vertical through G, the centre of gravity of ABC, fall within AC, meeting AC in E. Then we have

$$CF = a\cos(180° - C) = -a\cos C;$$

$$CD = \frac{b}{2}; \quad DE = \frac{1}{3}DF = \frac{1}{3}\left(\frac{b}{2} - a\cos C\right);$$

and
$$CE = CD - DE.$$

In the extreme case of equilibrium,

$$X \,.\, CF = W \,.\, CE,$$

i.e.
$$X(-a\cos C) = W\left(\frac{2}{3}\cdot\frac{b}{2} + \frac{1}{3}\cdot a\cos C\right),$$

$$\therefore\ X = \frac{W}{3}\cdot\frac{b + a\cos C}{-a\cos C} = \frac{W}{3}\cdot\left[b + \frac{a^2 + b^2 - c^2}{2b}\right] \div \left[\frac{c^2 - a^2 - b^2}{2b}\right]$$

$$= \frac{W}{3}\cdot\frac{a^2 + 3b^2 - c^2}{c^2 - a^2 - b^2}.$$

If $a^2 + 3b^2 - c^2$ be negative, i.e. $c^2 > a^2 + 3b^2$, CE is negative, i.e. the moment of W in the above sense is negative, and therefore the vertical through G falls without AC, so that the lamina would topple over even without a weight at B.

8. Let A, B, C, D .. be the extreme ends of the cards, A the top one; and let $2l$ be the length and W be the weight of each card. The top card can at most project a distance l beyond the next one, or it would slip off; i.e. its centre of gravity must be just supported by B. Then the centre of gravity of the top two cards is vertically over C; so that

$$W \,.\, BC = W(l - BC), \text{ and hence } BC = \frac{l}{2}.$$

So the centre of gravity of the top three cards is vertically over D;

$$\therefore\ 2W.CD = W(l - CD), \text{ and } CD = \frac{l}{3}.$$

And generally for n cards, we have

$$(n-1)W.x = W(l-x), \text{ and } x = \frac{l}{n}.$$

Thus the distances which each successive card projects over the next lower one are

$$l,\ \frac{l}{2},\ \frac{l}{3} \ldots\ldots \frac{l}{n},$$

and therefore form a harmonical progression.

9. By the triangle of forces the force aA is equivalent to forces aG and GA, so the force bB to bG and GB and so on.

Hence all the forces are equivalent to the set aG, bG,...... and the set AG, BG,......

The first set is by Ex. 12, p. 146, equivalent to $n.gG$, and the latter set is by the same example equivalent to zero.

If g coincide with G the original forces are in equilibrium.

10. If G be the original centre of gravity of the whole, g be the centre of gravity of the portion cut out, h be that of the remainder, and g move to g_1 so that $gg_1 = x$, then the centre of gravity of the whole is at G_1 in hg_1, such that

$$hG_1 : G_1g_1 = hG : Gg = w : W - w,$$

or G_1 moves along GG_1 which is parallel to gg_1; also

$$GG_1 : gg_1 = hG : hg = w : W,$$

$$\therefore\ GG_1 = \frac{w}{W}x.$$

As GG_1 is parallel to gg_1, the line joining the two positions of the centre of gravity of the whole body is parallel to the line joining the two positions of the centre of gravity of the portion moved.

11. Since $AC = \frac{1}{2}AB$, and the $\angle BAC = 60°$,

therefore the $\angle ACB = 90°$.

G is in DE, the line parallel to BC through D the middle point of AB, meeting AC in E; and, since

$$AB = 2AC,$$

we have $$DG : GE = 1 : 2,$$

so that $$DG = \frac{1}{3}DE = \frac{1}{3}\cdot\frac{1}{2}BC.$$

Draw DF and GH parallel to AC, meeting BC in F and H respectively. Then we have

$$BH=BF+FH=BF+DG=\left(\frac{1}{2}+\frac{1}{6}\right)BC$$
$$=\frac{2}{3}BC=\frac{2}{3}.AC\sqrt{3}=\frac{2}{\sqrt{3}}AC;$$

and
$$GH=CE=\frac{1}{2}AC;$$

$$\therefore\ BG=\sqrt{BH^2+GH^2}=AC\sqrt{\frac{4}{3}+\frac{1}{4}}=AC\sqrt{\frac{19}{12}}.$$

If when suspended from B, the action at A consist of a horizontal force X, a vertical force Y, and a couple K (the connection being rigid), resolving horizontally, we have $X=0$, and resolving vertically, we have

$$Y+\frac{W}{3}=0,$$

so that Y is a vertical force equal to $\frac{W}{3}$. [Since W is the weight of the system, we have $\frac{W}{3}$ acting at E the middle point of AC, and $\frac{2W}{3}$ acting at D the middle point of AB.] Also, moments about A give

$$K=\frac{W}{3}.AE\cos CBG=\frac{W}{3}.\frac{AC}{2}.\frac{BH}{BG}$$
$$=\frac{W}{6}.AC.\left(\frac{2}{\sqrt{3}}\div\sqrt{\frac{19}{12}}\right)=\frac{2}{3}W\frac{AC}{\sqrt{19}}.$$

12. If F be the required horizontal stress, since there must be an equal and opposite horizontal stress at the lower hinge, we have, by taking moments about the lower hinge,

$$F\times 4=500\times\frac{10}{2},$$

so that $F=625$ lbs. wt.

13. Let AB and AC be the two portions of the ladder, G being the middle point of AB, and let a weight W be placed at D where

$$BD=\frac{1}{n}.BA=\frac{2a}{n},$$

if the length of the ladder be $2a$. Let the weight of each portion of the ladder be W' and let the vertical reactions at B and C be R and S.

By resolving vertically and taking moments about B for the whole ladder, we have

$$R+S=W+2W',$$

and
$$2S=W\cdot\frac{1}{n}+W'\cdot\frac{1}{2}+W'\cdot\frac{3}{2}=\frac{W}{n}+2W'.$$

Hence
$$S=W'+\frac{W}{2n},$$

and
$$R=W'+W-\frac{W}{2n}.$$

Also, taking moments about A for the ladder AC, we have

$$T\cdot a\cos\alpha=S\cdot 2a\sin\alpha-W'a\sin\alpha;$$

$$\therefore\ T=\tan\alpha\,[2S-W']=\tan\alpha\left[W'+\frac{W}{n}\right].$$

If initially T_0 be the tension when there was no weight W on the ladder, we have, by putting W equal to zero,

$$T_0=\tan\alpha\,.\,W'.$$

Hence the increase in the tension

$$=T-T_0=\frac{1}{n}W\tan\alpha.$$

14. Let AB be the beam, A and B being its points of contact with the wall and the cylinder respectively; and let C be the centre of the cylinder and E its point of contact with the wall. Let S be the horizontal pressure on the wall at A, and R be the reaction at B. The directions of S, R and W meet in a point O. Let the vertical through G meet EC in D, and through B draw HBF vertical, meeting the horizontal from A in H, and EC in F; and let the $\angle FCB=\theta=$ the $\angle BOH$. Then we have

$$CF+EF=CE,$$

i.e.
$$r\cos\theta+2l\cos 45^\circ=r;$$

also
$$\tan\theta=\frac{BH}{OH}=2;$$

$$\therefore\ \frac{r}{\sqrt{5}}+\frac{2l}{\sqrt{2}}=r,\quad\text{and}\quad \frac{l}{r}=\frac{\sqrt{5}-1}{\sqrt{10}}.$$

Again, by Lami's Theorem, we have

$$\frac{R}{\sin\frac{\pi}{2}}=\frac{S}{\sin\left(\frac{\pi}{2}+\theta\right)}=\frac{W}{\sin(\pi-\theta)}.$$

$$\therefore\ R=\frac{W}{\sin\theta}=\frac{W\sqrt{5}}{2},$$

and
$$S=W\cot\theta=\frac{W}{2}.$$

15. Let the rod rest against the cylinder at A and B, where the pressures are R horizontal and S perpendicular to AB (B being an edge) respectively. Let G be the centre of gravity of the rod, W be its weight, and θ be its angle with the horizon. Also let W' be the weight of the cylinder, and E be the point of its base nearest to G. For equilibrium, the directions of R, S and W pass through a point O. We have

$$AO = AG\cos\theta = 16a\cos\theta,$$
$$AB = AO\cos\theta = 16a\cos^2\theta,$$
$$AC = AB\cos\theta = 16a\cos^3\theta,$$

C being the point where the horizontal through A cuts the cylinder,

i.e. $$2a = 16a\cos^3\theta.$$

$$\therefore \cos^3\theta = \frac{1}{8},$$

and $$\cos\theta = \frac{1}{2};$$

hence $$\theta = 60°.$$

In the extreme case of equilibrium, the moments about E must be equal,

$$\therefore W'.a = W.OC$$
$$= W(AO - AC) = W(16a\cos\theta - 2a)$$
$$= W(8a - 2a) = W.6a,$$

i.e. $$W' = 6W.$$

16. Let C be the centre of the sphere, G be the centre of gravity of the basin, O be the point where the axis meets CG (the axis being perpendicular to the plane of the paper), and W and w be the weights of the basin and the ball respectively. Suppose the basin displaced through a small angle θ; then, taking moments about O, we see that the basin will tip over if

$$w.CO\sin\theta > W.GO\sin\theta,$$

i.e. if $$w.c > W.a, \quad \textit{i.e.}\ \text{if}\quad w > \frac{a}{c}W.$$

17. Let A be the point of suspension, C be the centre of the base, CD be the radius perpendicular to the base, W be the weight of the water, and W' be the weight of the shell. The area of the shell $=2\pi r^2$, r being the radius; the area of the base $=\pi r^2$; the centre of gravity of the shell is at the middle point of CD [Art. 120]; the centre of gravity of the base is at C. Hence, if K be the centre of gravity of the shell and base, we have

$$CK = \frac{2\pi r^2 . \frac{r}{2} + \pi r^2 . 0}{2\pi r^2 + \pi r^2} = \frac{r}{3}.$$

Also, if H be the centre of gravity of the water,

$$CH=\frac{3}{8}r, \quad \text{[Art. 117]}.$$

Again, if G be the centre of gravity of the whole, G is vertically below A, and hence

$$CG=r\tan\alpha.$$

Also
$$CG=\frac{W\cdot\frac{3}{8}r+W'\cdot\frac{r}{3}}{W+W'}.$$

$$\therefore\ \frac{\frac{3}{8}W+\frac{1}{3}W'}{W+W'}=\tan\alpha.$$

$$\therefore\ W\left(\frac{3}{8}-\tan\alpha\right)=W'\left(\tan\alpha-\frac{1}{3}\right),$$

i.e.
$$W:W'=\tan\alpha-\frac{1}{3}:\frac{3}{8}-\tan\alpha.$$

18. Draw a figure representing a vertical section of the system through the centres of the spheres. Let R be the reaction between the cylinder and either sphere, [for, by symmetry, or by equating horizontal components, we see that the reactions are the same on each side], S be the reaction between the two spheres, and θ be the inclination of the distance between the centres of the spheres to the horizon. Then, for either sphere, resolving horizontally, we have

$$R=S\cos\theta;$$

and resolving vertically, we have,

$$W'=S\sin\theta;$$

$$\therefore\ \frac{R}{W'}=\cot\theta, \quad i.e.\ \ R=W'\cot\theta.$$

Taking moments about A, the point at which the reaction of the ground acts, for equilibrium we have

$$R\cdot r+W\cdot a=R\,(r+2r\sin\theta);$$

$$\therefore\ W\cdot a=2R\cdot r\sin\theta=2W'\cdot r\cos\theta;$$

but
$$\cos\theta=\frac{2a-2r}{2r}=\frac{a-r}{r};\ \text{hence}\ \ W\cdot a=2W'(a-r).$$

19. Let ACB be the lamina, the vertical angle C being 2α; also let P and Q be the pegs under the sides AC and BC respectively, D be the middle point of the base AB, and θ be the inclination of AB to the vertical. The directions of the reactions at P and Q (which are perpendicular to AC and BC) must meet the direction of W the weight of the board, which acts vertically through G its centre of gravity, in a point O. A circle, diameter CO, will go round $POQC$,

in which circle the chord PQ subtends an angle 2α at the circumference;

$$\therefore\ PQ = CO \sin 2\alpha \ \ldots \qquad \ldots(1).$$

Also we have the $\angle PCO =$ the $\angle PQO$,

in the same segment,

$$= \text{the } \angle PQB - \frac{\pi}{2} = \theta + \alpha - \frac{\pi}{2};$$

$$\therefore\ \text{the } \angle GCO = \alpha - \left(\theta + \alpha - \frac{\pi}{2}\right) = \frac{\pi}{2} - \theta = \text{the } \angle OGC;$$

$$\therefore\ GO = OC,$$

and the $\angle GOC = 2\theta$;

also $$\frac{CO}{CG} = \frac{\sin CGO}{\sin COG} = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin 2\theta} = \frac{\cos\theta}{2\sin\theta\cos\theta} = \frac{1}{2\sin\theta},$$

hence, by (1), $$PQ = \frac{CG}{2\sin\theta} \cdot \sin 2\alpha = CG\,\frac{\sin\alpha\cos\alpha}{\sin\theta};$$

now $$PQ = \frac{1}{3}AB,$$

and $$CG = \frac{2}{3}CD = \frac{2}{3} \cdot \frac{1}{2}AB\cot\alpha,$$

$$\therefore\ \frac{1}{3}AB = \frac{1}{3}AB\cot\alpha \cdot \frac{\sin\alpha\cos\alpha}{\sin\theta}.$$

$$\therefore\ \sin\theta = \cos^2\alpha, \quad \textit{i.e. } \theta = \sin^{-1}(\cos^2\alpha).$$

20. Let ABC be the principal equilateral section, A and B being the points of contact with the planes inclined at α and β respectively (B lower than A). Let D be the centre of AB, G be the centre of gravity of the prism, R and S be the reactions of the planes at A and B respectively, and θ be the inclination of AB to the vertical. Then, resolving horizontally, we have

$$R\sin\alpha = S\sin\beta.$$

Also, moments about G give

$$R\,[AD\cos(\alpha - 90^\circ + \theta) - GD\sin(\alpha - 90^\circ + \theta)]$$
$$= S\,[BD\cos(\beta + 90^\circ - \theta) + GD\sin(\beta + 90^\circ - \theta)],$$

i.e. $$R\left[\sin(\theta + \alpha) + \frac{1}{\sqrt{3}}\cos(\theta + \alpha)\right] = S\left[\sin(\theta - \beta) - \frac{1}{\sqrt{3}}\cos(\theta - \beta)\right].$$

$$\therefore\ \sin\beta\,[\sqrt{3}\sin(\theta + \alpha) + \cos(\theta + \alpha)] = \sin\alpha\,[\sqrt{3}\sin(\theta - \beta) - \cos(\theta - \beta)]$$

$$\therefore\ \tan\theta = \frac{\sin\beta\cos\alpha + \sin\beta\sin\alpha\sqrt{3} + \sin\alpha\sin\beta\sqrt{3} + \sin\alpha\cos\beta}{-\sqrt{3}\sin\beta\cos\alpha + \sin\beta\sin\alpha + \sqrt{3}\sin\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{\sin(\alpha + \beta) + 2\sqrt{3}\sin\alpha\sin\beta}{\sqrt{3}\sin(\alpha - \beta)}.$$

If θ be reckoned positive, whether $\alpha >$ or $< \beta$, this may be written

$$\tan\theta = \frac{\sin(\alpha+\beta) + 2\sqrt{3}\sin\alpha\sin\beta}{\sqrt{3}\sin(\alpha\sim\beta)};$$

this sign shews also that B is lower than A only if $\alpha > \beta$, and vice versâ.

Otherwise thus: Let the directions of R, S and W the weight of the prism meet in the point O, and let the vertical through G meet AB in E. Then we have

$$\frac{EO}{BE} = \frac{\sin(\theta-\beta)}{\sin\beta}, \text{ and } \frac{EO}{AE} = \frac{\sin(\theta+\alpha)}{\sin\alpha};$$

hence, by division, $\dfrac{BE}{AE} = \dfrac{\sin(\theta+\alpha)}{\sin\alpha} \cdot \dfrac{\sin\beta}{\sin(\theta-\beta)}$,

i.e. $$\frac{BE}{AE} = \frac{\cot\alpha + \cot\theta}{\cot\beta - \cot\theta}.$$

If $$AB = a, \text{ then } CD = \frac{a\sqrt{3}}{2},$$

and $$GD = \frac{1}{3} \cdot \frac{a\sqrt{3}}{a} = \frac{a}{2\sqrt{3}};$$

$$\therefore\ DE = \frac{a\cot\theta}{2\sqrt{3}}.$$

Hence.

$$\frac{BE}{AE} = \frac{BD - DE}{AD + DE} = \frac{\dfrac{a}{2} - \dfrac{a\cot\theta}{2\sqrt{3}}}{\dfrac{a}{2} + \dfrac{a\cot\theta}{2\sqrt{3}}} = \frac{\sqrt{3} - \cot\theta}{\sqrt{3} + \cot\theta} = \frac{\cot\alpha + \cot\theta}{\cot\beta - \cot\theta};$$

$$\therefore\ \sqrt{3}(\cot\beta - \cot\alpha) = \cot\theta(\cot\beta + \cot\alpha + 2\sqrt{3}),$$

$$\therefore\ \tan\theta = \frac{2\sqrt{3} + \cot\beta + \cot\alpha}{\sqrt{3}(\cot\beta - \cot\alpha)} = \frac{2\sqrt{3}\sin\alpha\sin\beta + \sin(\alpha+\beta)}{\sqrt{3}\sin(\alpha-\beta)},$$

or $\sin(\beta - \alpha)$ in the denominator if $\beta > \alpha$.

21. Let l be the length of the string, T be its tension, θ be its inclination to the horizon, h be the height of the triangle, a be its side, and x be the distance from the edge of the table of the point on it to which the string is attached. Then, taking moments about the edge of the table, we have

$$1 \cdot \frac{a}{4} = T \cdot x\sin\theta; \quad \text{and} \quad \sin\theta = \frac{h}{l} = \frac{1}{2}, \text{ i.e. } \theta = 30°;$$

also $$x \cdot \frac{a}{4} = h\cot\theta = \frac{a\sqrt{3}}{2} \cdot \sqrt{3} = \frac{3a}{2}.$$

$$\therefore\ x = \frac{3a}{2} - \frac{a}{4} = \frac{5a}{4}; \text{ hence } T = \frac{a}{4} \div \left[\frac{5a}{4} \cdot \frac{1}{2}\right] = \frac{2}{5} \text{ lb wt.}$$

22. For equilibrium, the lines of action of W the weight of the cone, R the resultant normal reaction of the wall, and T the tension of the string must meet in a point O. Since R is horizontal, and cannot act higher than through A, the upper edge of the base AB, it follows that the string is longest when the cone is on the point of turning round A. Let C be the vertex of the cone, D be the centre of AB, G be its centre of gravity and F be the point of attachment of the string to the wall. Then we have

$$DG=\frac{h}{4}=AO, \text{ and } DA=h\tan\alpha;$$

also, by similar triangles FAO and FDC, we have $FA=\frac{1}{4}FD$.

Hence, if $$FA=x,$$

we have $$x=\frac{1}{4}(x+h\tan\alpha), \text{ i.e. } x=\frac{h}{3}\tan\alpha,$$

and hence $$FD=\frac{4}{3}h\tan\alpha;$$

thus $$CF^2=\sqrt{CD^2+FD^2}=h\sqrt{1+\frac{16}{9}\tan^2\alpha}.$$

Otherwise thus: If the string be at an angle θ to the vertical, we have $$W=T\cos\theta,$$

and moments about A give

$$W\cdot\frac{h}{4}=T(h\cot\theta-h\tan\alpha)\sin\theta.$$

$$\therefore\ \frac{1}{4}\cos\theta=\cos\theta-\sin\theta\tan\alpha, \text{ whence } \cot\theta=\frac{4}{3}\tan\alpha;$$

and the length of the string required

$$=h\operatorname{cosec}\theta=h\sqrt{1+\cot^2\theta}=h\sqrt{1+\frac{16}{9}\tan^2\alpha}.$$

23. Let A be the vertex of the cone, B be the centre of its base, G be its centre of gravity, D be the peg, and C be the point on the circumference of the base to which the string is fastened. The peg being smooth, the tension of the string is the same throughout, and since the equal tensions along CD and AD meet in D, the line of action of W, the weight of the cone, must pass through D; hence G must be vertically below D. Also, since W balances the resultant of two equal forces, GD must bisect the angle between them, *i.e.* the angle CDA. Let DG meet CA in E, and CD and DA be denoted by x and y respectively. Then, by Geometry, we have

$$\frac{CE}{EA}=\frac{BG}{GA}=\frac{h}{4}\div\frac{3h}{4}=\frac{1}{3}, \text{ and } \frac{x}{y}=\frac{CE}{EA}=\frac{1}{3};$$

hence $$x=\frac{l}{4}, \text{ and } y=\frac{3l}{4},$$

where l is the length of the string.

Equating the two values of DG, we have

$$\sqrt{y^2-\left(\frac{3h}{4}\right)^2}=r+\sqrt{x^2-\left(\frac{h}{4}\right)^2},$$

$$\therefore \frac{3}{4}\sqrt{l^2-h^2}=r+\frac{1}{4}\sqrt{l^2-h^2}.$$

$$\therefore \sqrt{l^2-h^2}=2r.$$

$$\therefore l^2=h^2+4r^2, \text{ so that } l=\sqrt{h^2+4r^2}.$$

24. The centres of the four spheres form a regular tetrahedron, and if d be the diameter of any one of the spheres, A be the centre of the upper one, B be the centre of a lower one, and G be the centre of gravity of the triangle formed by the centres of the three lower ones, then AG is clearly vertical, and the angle ABG $(=\theta)$ is the angle made by AB with the horizon, and AB is the direction of the mutual action R between the spheres centre A and centre B. Thus, if W be the weight of the upper sphere, we have

$$W=3R\sin\theta.$$

Also, the string forms three straight parts of an equilateral triangle, and therefore resolving along BG for the sphere B, if T be the tension, we have

$$2T\,.\cos 30°=R\cos\theta.$$

$$\therefore T\sqrt{3}=\frac{W}{3}\cot\theta.$$

Now $$BG=\frac{2}{3}d\,\frac{\sqrt{3}}{2}=\frac{d}{\sqrt{3}}, \text{ and } AB=d;$$

$$\therefore \cos\theta=\frac{1}{\sqrt{3}}, \text{ and } \sin\theta=\frac{\sqrt{2}}{\sqrt{3}};$$

hence $$T=\frac{W}{3\sqrt{3}}\cdot\frac{1}{\sqrt{2}}=\frac{W}{3\sqrt{6}}, \text{ i.e. } T:W=1:3\sqrt{6}.$$

25. Let the rod rest against the plane at the point A; let B represent the rail, BL $(=c)$ be a perpendicular on the plane, and R and S be the pressures at B, perpendicular to AB, and at A, perpendicular to the plane respectively; also let W be the weight of the rod and BA make an angle θ with the plane. Then, resolving along the plane, we have

$$W\sin\alpha=R\sin\theta.$$

Also, taking moments about A, we have

$$W\,.\,a\cos(\theta-\alpha)=R\,.\,AB=R\,.\,c\operatorname{cosec}\theta.$$

Hence $$a\sin^2\theta\cos(\theta-\alpha)=c\sin\alpha.$$

Otherwise thus: The directions of R, S and W meet in a point O. We have

$$AB = c \operatorname{cosec} \theta,$$

and $$AO = AB \sec BAO = AB \operatorname{cosec} \theta = c \operatorname{cosec}^2 \theta;$$

now $$\frac{AO}{AG} = \frac{\sin AGO}{\sin AOG} = \frac{\cos(\theta - \alpha)}{\sin \alpha};$$

$$\therefore \frac{c \operatorname{cosec}^2 \theta}{a} = \frac{\cos(\theta - \alpha)}{\sin \alpha}.$$

$$\therefore a \sin^2 \theta \cos(\theta - \alpha) = c \sin \alpha.$$

26. Let $ABCD$ be the board, AB being the upper edge, and APB be the string. Let G be the centre of gravity of the board, and describe a circle round AGB.

The locus of P is an ellipse, foci A and B; and, since the peg is smooth, the tension of the string is the same throughout; also the direction of the weight along PG bisects the angle APB. The position of symmetry (*i.e.* with AB horizontal) is one of equilibrium, and if the ellipse cut the circle round AGB in two points, Q and Q', we have two other positions of equilibrium; QG and $Q'G$ will be vertical, and the $\angle AQG$ = the $\angle BQG$, since arc AG = arc BG; also the $\angle AQ'G$ = the $\angle BQ'G$. The condition that the ellipse should cut the circle is

$$AP + PB < AG + GB,$$

i.e. the length of the string < the diagonal of the board.

27. [Cf. the figure on p. 89, but the bowl turned round, with C much lower down.] Let ACB be the rod resting in the bowl at A and on the rim at C. Let O be the centre of the base of the bowl, and OL be the radius perpendicular to the base; then, if G' be the centre of gravity of the bowl,

$$OG' = \frac{1}{2} OL = \frac{r}{2} \text{ [Art. 117]}.$$

Let R be the reaction at A along AO, and S be the reaction at C perpendicular to the rod. These two reactions meet in a point D; also, by Euc. III. 31, D must lie on the geometrical sphere of which the bowl is a portion. Hence the vertical line through G, the middle point of the rod, must pass through D. The

$$\angle AOC = 2\beta.$$

$$\therefore \text{the } \angle ADC = \beta.$$

Let F be the point of contact of the bowl with the table; since the weights of the bowl and the rod are equal, moments about F are equal, *i.e.* $KF = FH$, K and H being the points where the vertical lines through G' and G respectively meet the table; but $AO = OD$; $\therefore G'K$ passes through A. Through O and A draw ON and AE horizontal to meet DH in N and E respectively.

The $\angle DON = 180° - (\alpha + 2\beta) =$ the $\angle OAE$; the $\angle OAC = 90° - \beta$;

$$\therefore \text{ the } \angle GAE = 90° - \alpha - \beta.$$

Now $\quad AE = AD \cos DAE$

$$= 2r \cos [180° - (\alpha + 2\beta)] = -2r \cos (\alpha + 2\beta);$$

also, $\quad AE = AG \cos GAE = l \cos (90° - \alpha - \beta) = l \sin (\alpha + \beta)$;

again, $\quad KF = OG' \sin \alpha = \frac{r}{2} \sin \alpha.$

$$\therefore AE = 2KF = r \sin \alpha;$$

hence we have

$$l \sin (\alpha + \beta) = r \sin \alpha = -2r \cos (\alpha + 2\beta).$$

28. If the strings make an angle θ with the wall, and their tensions each be equal to T, and R be the mutual action of the rod and the wedge, perpendicular to the face of the latter in contact with the rod, then resolving vertically for the wedge, we have

$$R \cos 60° = \frac{W}{2}.$$

$$\therefore R = W.$$

Also, resolving horizontally and vertically for the rod, we have

$$2T \sin \theta = R \cos 30° = \frac{W\sqrt{3}}{2},$$

and

$$2T \cos \theta = R \cos 60° + W = \frac{3W}{2}.$$

Hence, by division, $\quad \tan \theta = \frac{1}{\sqrt{3}}$, *i.e.* $\theta = 30°$,

and the rod is thrust from the wall through the distance $= l \sin \theta = \frac{l}{2}$.

29. If R be the pressure between the plank and the cylinder, for the equilibrium of the plank, taking moments about A, we have

$$R \cdot r \cot \frac{\theta}{2} = W \cdot a \cos (\alpha + \theta) \quad \dots\dots\dots\dots\dots (1),$$

and resolving along the plane, we have

$$R \sin \theta = W' \sin \alpha \quad \dots\dots\dots\dots\dots\dots (2).$$

Eliminating R between (1) and (2), we have

$$W' \cdot r \cdot \frac{\sin \alpha}{\sin \theta} \cdot \cot \frac{\theta}{2} = W \cdot a \cos (\alpha + \theta).$$

$$\therefore \frac{W'r}{Wa} = \frac{\cos (\alpha + \theta)}{\sin \alpha} \cdot \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cot \frac{\theta}{2}} = \frac{\cos (\alpha + \theta)}{\sin \alpha} \cdot 2 \sin^2 \frac{\theta}{2}$$

$$= \cos (\alpha + \theta) \frac{1 - \cos \theta}{\sin \alpha}.$$

30. Since the two equal discs are similarly situated, it is only necessary to consider the equilibrium of one of them. If A be the centre of either disc, and B be the point in which it touches a plane, then the least disc required is that which touches the former at C in BA produced, since any smaller disc would give rise to a pressure through A having a component perpendicular to AB and outwards in direction, and thus the disc, centre A, would move out; and so of course would the other disc, symmetrically. If, then, O be the point in which BA produced meets the bisector of the angle 2α, O is the centre of the least disc required, and its radius is OC; also

$$OC + CA = r \sec \alpha,$$

so that

$$OC = r(\sec \alpha - 1).$$

31. Let $AB = CD = a$, and $AD = BC = b$; also $AP = x$. Let W be the weight placed at P, T be the tension along CA, and let the angle ACB be θ.

Then the upward thrust in $BC = T \cos \theta =$ the pressure at B. Taking moments about A, for the equilibrium of AB, we have

$$T \cos \theta \times AB = W \times x, \quad i.e. \quad T \frac{BC \times AB}{AC} = Wx.$$

$$\therefore \ T = W \frac{x \sqrt{a^2 + b^2}}{ab}.$$

If W act at E on CD, vertically below P, and moments be taken about D, we have the same result for the equilibrium of CD.

32. Let θ be the inclination of MN to the horizon, and R and S be the normal reactions at M and N respectively. Then, resolving horizontally, we have

$$\mu R \cos \alpha - R \sin \alpha + \mu S \cos \beta + S \sin \beta = 0;$$

$$\therefore \ R(\sin \alpha - \cos \alpha \tan \epsilon) = S(\sin \beta + \cos \beta \tan \epsilon),$$

i.e.

$$R \sin(\alpha - \epsilon) = S \sin(\beta + \epsilon) \ \ldots\ldots\ldots\ldots\ldots \ldots (1).$$

Also, taking moments about the centre of the rod, we have

$$\mu R \sin(\alpha - \theta) + R \cos(\alpha - \theta) = S \cos(\theta + \beta) - \mu S \sin(\theta + \beta),$$

i.e.

$$R \cos(\alpha - \epsilon - \theta) = S \cos(\beta + \epsilon + \theta),$$

whence

$$\tan \theta = \frac{-R \cos(\alpha - \epsilon) + S \cos(\beta + \epsilon)}{R \sin(\alpha - \epsilon) + S \sin(\beta + \epsilon)};$$

hence, by (1), we have

$$\tan \theta = \frac{-\sin(\beta + \epsilon) \cos(\alpha - \epsilon) + \sin(\alpha - \epsilon) \cos(\beta + \epsilon)}{\sin(\alpha - \epsilon) \sin(\beta + \epsilon) + \sin(\beta + \epsilon) \sin(\alpha - \epsilon)}$$

$$= \frac{\sin(\alpha - \beta - 2\epsilon)}{2 \sin(\beta + \epsilon) \sin(\alpha - \epsilon)}.$$

Otherwise thus: The reactions at M and N act at angles ϵ with the normals. Let the lines of action of R and S meet in the point O; then O is vertically above G, the middle point of MN.

Draw ME and NF horizontal, meeting OG in E and F respectively. Then we have

$$ME = NF, \text{ and } EG = GF;$$

$$\therefore\ 2EG = OF - OE,$$

i.e.
$$2ME \,.\, \tan\theta = NF\cot(\beta+\epsilon) - ME\cot(\alpha-\epsilon);$$

$$\therefore\ 2\tan\theta = \cot(\beta+\epsilon) - \cot(\alpha-\epsilon).$$

$$\therefore\ \tan\theta = \frac{1}{2}\,.\,\frac{\sin[(\alpha-\epsilon)-(\beta+\epsilon)]}{\sin(\beta+\epsilon)\sin(\alpha-\epsilon)} = \frac{\sin(\alpha-\beta-2\epsilon)}{2\sin(\beta+\epsilon)\sin(\alpha-\epsilon)}.$$

33. If AB be the rod, on the point of slipping round A, every point of it is on the point of moving perpendicular to AB; therefore the friction at every point is limiting and perpendicular to AB, and R the resultant normal pressure of the weight W on the plane $= W\cos\alpha$, at the middle point of the rod, and therefore the resultant friction will act through that point also, and

$$= R\tan\lambda = W\cos\alpha\,.\,\tan\lambda.$$

Also the component of the weight parallel to the plane $= W\sin\alpha$, parallel to the line of greatest slope; hence, if this make an angle θ with AB, moments about A give

$$W\cos\alpha\,.\,\tan\lambda\,.\,\frac{AB}{2} = W\sin\alpha\,.\,\frac{AB}{2}\,.\,\sin\theta.$$

$$\therefore\ \sin\theta = \tan\lambda\cot\alpha,\quad i.e.\ \theta = \sin^{-1}(\tan\lambda\cot\alpha).$$

The same result clearly would follow if B were lower than A, and θ is possible, since $\tan\lambda < \tan\alpha$, and therefore $\tan\lambda\cot\alpha < 1$.

34. The action at the hinge is by symmetry horizontal, and if θ be the inclination of either rod to the vertical, W be its weight, and R and μR be the normal and frictional reactions between it and the sphere, if the friction act upwards, resolving vertically for either rod, we have

$$W = R\sin\theta + \mu R\cos\theta;$$

also, taking moments about the hinge, we have

$$R\,.\,c\cot\theta = W\,.\,a\sin\theta;$$

hence
$$c\cot\theta = a\sin\theta(\sin\theta + \mu\cos\theta);$$

if $\mu = \dfrac{c}{a}$, the limiting position is given by

$$c\cot\theta = a\sin^2\theta + c\sin\theta\cos\theta,$$

i.e.
$$c\cos\theta = a\sin^3\theta + c\sin^2\theta\cos\theta,$$

and
$$\therefore\ a\sin^3\theta = c\cos^3\theta,$$

and
$$\tan\theta = \sqrt[3]{c \div a},\quad i.e.\ \theta = \tan^{-1}\sqrt[3]{c \div a}.$$

This gives the least value which θ can have, but if it be sought to increase θ considerably, the hinge will be forced up, and the friction then act downwards, and the greatest value of θ will correspond to a root of the equation

$$c \cot\theta = a \sin^2\theta - c \sin\theta \cos\theta, \quad \text{or} \quad a \sin^3\theta = c \cos\theta\,(1 + \sin^2\theta).$$

35. If AB be the rod, C be its middle point, and O be the centre of the sphere, $OC = \sqrt{a^2 - c^2} =$ a constant, and, therefore, if the rod be on the point of sliding down, it must tend to turn parallel to itself round O, as on a pendulum OC; therefore the frictions at A and B are perpendicular to the plane AOB, which is at an angle θ to the vertical plane through AB, where θ is the inclination to the vertical of OC. Also, if R be the normal reaction at A, it acts along AO, and the component of R along CO

$$= R \,.\, \frac{\sqrt{a^2 - c^2}}{a};$$

the other component, perpendicular to CO, is neutralised by that corresponding to B. Hence, resolving in a horizontal direction at right angles to the rod, we have

$$R \,.\, \frac{\sqrt{a^2 - c^2}}{a} \,.\, \sin\theta = \mu R \cos\theta, \quad \text{whence} \quad \theta = \tan^{-1} \frac{\mu a}{\sqrt{a^2 - c^2}}.$$

36. If O be the fixed point, OL be the perpendicular on the fixed rod, X be the farthest point of this rod at which the movable one can rest, then the required portion $= 2LX$, and R, the normal reaction, will be perpendicular to both rods, and therefore to the plane XOL, while the friction is opposite to the direction in which the point X on the movable rod is on the point of moving, and therefore perpendicular to OX in the plane XOL. This plane makes an angle α with the horizon, and the component of the weight W at G parallel to OL

$$= W \sin\alpha;$$

hence moments about the perpendicular to this plane through O give

$$\mu R \,.\, OX = W \,.\, OG \sin XOL \,.\, \sin\alpha;$$

also moments about the perpendicular to OX through O in the plane XOL give

$$R \,.\, OX = W \cos\alpha \,.\, OG;$$

$$\therefore\ \mu \cos\alpha = \sin XOL \,.\, \sin\alpha;$$

also $$\tan XOL = \frac{LX}{b}; \text{ hence } \frac{LX}{b} = \frac{\mu \cot\alpha}{\sqrt{1 - \mu^2 \cot^2\alpha}},$$

and the required portion

$$= 2LX = \frac{2\mu b \cos\alpha}{\sqrt{\sin^2\alpha - \mu^2 \cos^2\alpha}}.$$

37. Let W be the weight of the rod, a be the distance of its centre of gravity from A its lower end, d be the diameter of the tumbler, μ be the coefficient of friction, R be the horizontal reaction on A from the tumbler, and S be the normal reaction between the rod and the rim at C. The rod makes angles α and β respectively with the vertical when it is on the point of slipping *up* and *down* the side of the tumbler. For the equilibrium of the rod, we have

$$W+\mu S\cos\alpha-S\sin\alpha+\mu R=0,$$

and
$$R=\mu S\sin\alpha+S\cos\alpha;$$

also moments about A give

$$W.\,a\sin\alpha=Sd\operatorname{cosec}\alpha.$$

Thus we have

$$W+S(\mu\cos\alpha-\sin\alpha+\mu^2\sin\alpha+\mu\cos\alpha)=0,$$

and
$$\therefore\ d+a\sin^2\alpha[2\mu\cos\alpha-(1-\mu^2)\sin\alpha]=0,$$

so that, if
$$\mu=\tan\lambda,$$

we have $\quad d\cos^2\lambda+a\sin^2\alpha\,(\cos\alpha\sin 2\lambda-\sin\alpha\cos 2\lambda)=0.$

Similarly, when λ is negative, this equation holds with β written for α;

$$\therefore\ d\cos^2\lambda+a\sin^2\beta\,(-\cos\beta\sin 2\lambda-\sin\beta\cos 2\lambda)=0.$$

Hence
$$\sin^2\alpha\,(\sin\alpha\cos 2\lambda-\cos\alpha\sin 2\lambda)$$
$$=\sin^2\beta\,(\sin\beta\cos 2\lambda+\cos\beta\sin 2\lambda).$$

$$\therefore\ \tan 2\lambda=\frac{\sin^3\alpha-\sin^3\beta}{\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta},$$

i.e.
$$\lambda=\frac{1}{2}\tan^{-1}\frac{\sin^3\alpha-\sin^3\beta}{\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta}.$$

Otherwise thus: Let ACB be the rod, resting in the tumbler at A and on the rim at C, G be the centre of the rod, and W be its weight. Draw AD horizontal through A, meeting the opposite side of the tumbler in D; and draw CE perpendicular to the rod at C. The rod makes angles α and β respectively with the vertical, when it is on the point of slipping *up* and *down* the side of the tumbler.

(1) When the rod is on the point of slipping *up*. The directions of R the reaction at A, S the reaction at C, and W meet in a point O, below AD; and the $\angle DAO=\lambda=$ the angle the direction of S makes with CE. Then we have

$$\frac{AO}{AG}=\frac{\sin\alpha}{\sin(90^\circ-\lambda)}=\frac{\sin\alpha}{\cos\lambda},$$

$$\frac{AC}{AO}=\frac{\sin(\alpha-2\lambda)}{\sin(90^\circ+\lambda)}=\frac{\sin(\alpha-2\lambda)}{\cos\lambda},\ \text{and}\ \frac{AD}{AC}=\sin\alpha;$$

hence, by multiplication,

$$\frac{AD}{AG}=\frac{\sin^3\alpha\sin(\alpha-2\lambda)}{\cos^2\lambda}\ \ \text{.........\ ....\ ..(1.)}$$

(2) When the rod is on the point of slipping *down.* O is then above AD, and writing β for α, and $-\lambda$ for λ in (1), we have

$$\frac{AD}{AG}=\frac{\sin^2\beta\sin(\beta+2\lambda)}{\cos^2\lambda} \quad\text{.....................}(2).$$

From (1) and (2), we have

$$\sin^2\alpha\sin(\alpha-2\lambda)=\sin^2\beta\sin(\beta+2\lambda).$$

$$\therefore\ \sin^3\alpha\cos 2\lambda-\sin^2\alpha\cos\alpha\sin 2\lambda=\sin^3\beta\cos 2\lambda+\sin^2\beta\cos\beta\sin 2\lambda.$$

$$\therefore\ \sin^3\alpha-\sin^3\beta=(\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta)\tan 2\lambda,$$

whence
$$\lambda=\frac{1}{2}\tan^{-1}\frac{\sin^3\alpha-\sin^3\beta}{\sin^2\alpha\cos\alpha+\sin^2\beta\cos\beta}.$$

38. The rod rests in a plane perpendicular to the side and the edge of the box. Let $LEDC$ be the section of the box by this plane, ED being the base. Let ACB be the rod, resting in the box at A and on the edge at C (C being the point over D). Let G be the centre of the rod and W be its weight; then the weight of the box is $4W$. Let R and S be the reactions at A and C respectively; through A draw AKM horizontal, meeting CD in K and the vertical through G in M; let ED produced meet the vertical through G in F, and let the line of action of $4W$ meet ED in H, and θ be the angle the rod makes with the vertical. Since the system is about to turn round D, we have

$$4W.DH=W.DF.$$

$$\therefore\ DF=4DH=2DE,$$

and
$$AM=3AK.$$

The directions of the three forces on the rod, R, S and W, pass through O, O being vertically below G, and

$$\text{the } \angle OAK=\lambda.$$

Thus
$$AM=AO\cos\lambda,$$

and

$$AK=AC\sin\theta=AO\cos(\lambda+90^\circ-\theta)\sin\theta=AO\sin(\theta-\lambda)\sin\theta.$$

$$\therefore\ \cos\lambda=3\sin(\theta-\lambda)\sin\theta.$$

$$\therefore\ 2\cos\lambda=3[\cos\lambda-\cos(2\theta-\lambda)].$$

$$\therefore\ 3\cos(2\theta-\lambda)=\cos\lambda.$$

$$\therefore\ \cos(2\theta-\lambda)=\frac{1}{3}\cos\lambda.$$

$$\therefore\ 2\theta-\lambda=\cos^{-1}\left(\frac{1}{3}\cos\lambda\right),\ \textit{i.e.}\ \theta=\frac{1}{2}\lambda+\frac{1}{2}\cos^{-1}\left(\frac{1}{3}\cos\lambda\right).$$

Otherwise thus: Let $2a$ be the width of the box. Taking moments about D for the whole system, we have

$$W.CG\sin\theta=4W.a\text{.............................}(1).$$

Also, resolving along the rod, and taking moments about C, for the rod, we have

$$W\cos\theta = R(\sin\theta - \mu\cos\theta),$$

and $$R(2a\cot\theta + \mu . 2a) = W . CG\sin\theta = 2W . a, \text{ by (1)};$$

$$\therefore\ (\cot\theta + \mu)\cos\theta = 2(\sin\theta - \mu\cos\theta).$$

$$\therefore\ \cos^2\theta + 3\mu\sin\theta\cos\theta - 2\sin^2\theta = 0.$$

$$\therefore\ (1+\cos 2\theta) + 3\mu\sin 2\theta - 2(1-\cos 2\theta) = 0,$$

or, if $2\theta = \phi$, $$3\mu\sin\phi + 3\cos\phi = 1.$$

$$\therefore\ \frac{1}{3}\cos\lambda = \cos(\phi - \lambda).$$

$$\therefore\ \phi = \lambda + \cos^{-1}\left(\frac{1}{3}\cos\lambda\right),$$

$$\theta = \frac{1}{2}\lambda + \frac{1}{2}\cos^{-1}\left(\frac{1}{3}\cos\lambda\right).$$

39. Let AB be the rod pulled at C ($AC < CB$) with a horizontal force P perpendicular to AB, and G be the centre of gravity of the rod. Then C will begin to move perpendicular to AB; therefore the rod's centre of instantaneous rotation is some point O in its length. If O were in AB produced, all the frictions on the elements of the rod would act in the same sense perpendicular to AB;

$$\therefore\ P = \mu W,$$

W being the weight of the rod; also, moments about C would give

$$P . OC = \mu W . OG,$$

which equations do not agree unless C be at G, and then they leave O indeterminate. Similarly, O cannot be in BA produced, and must therefore be in AB. Then for points in BO the frictions act in the same sense as P, and for points in OA in the opposite sense. Hence, resolving perpendicular to AB, and taking moments about O, we have, if

$$OG = x, \quad GC = a, \quad \text{and } AB = 2c,$$

$$P + \mu W . \frac{c-x}{2c} = \mu W . \frac{c+x}{2c}, \quad i.e. \quad P = \mu W . \frac{x}{c},$$

and $$\mu W . \frac{c-x}{2c} . \frac{c-x}{2} + \mu W . \frac{c+x}{2c} . \frac{c+x}{2} = P(a+x).$$

$$\therefore\ \frac{\mu W}{4c}[(c-x)^2 + (c+x)^2] = \frac{\mu W}{c} . x(a+x).$$

$$\therefore\ \frac{x^2+c^2}{2} = x(a+x).$$

$$\therefore\ x^2 + 2ax + a^2 = c^2 + a^2,$$

and $$x = \sqrt{c^2+a^2} - a,$$

since the other sign of the radical would place O in BA produced.

If P be P_1 and P_2 when $a=0$, and $a=c$, respectively, and, therefore, $x=c$, and $x=c\sqrt{2}-c$, respectively,

we have $$P_1=\mu W \cdot \frac{c}{c}=\mu W,$$

and $$P_2=\mu W \cdot \frac{c(\sqrt{2}-1)}{c}=\mu W(\sqrt{2}-1);$$

hence $$P_1 : P_2=1 : \sqrt{2}-1=\sqrt{2}+1 : 1.$$

40. If A and B be the particles, each of weight W, C and D the points of attachment of the strings, E the middle point of the rod, μ the coefficient of friction when the rod is on the point of turning round E, R the normal reaction at A or B, and if T be the tension of either string, then, resolving vertically, we have

$$2R+2T\cos\theta=2W,$$

i.e. $$R=W-T\cos\theta;$$

also, moments about E in the horizontal plane give

$$2T\sin\theta \cdot a=2\mu R \cdot c,$$

since, the rod tending to turn round E, the frictions are perpendicular to AB. Thus we have

$$T \cdot a\sin\theta=\mu c(W-T\cos\theta).$$

$$\therefore\ T=\frac{\mu c W}{a\sin\theta+\mu c\cos\theta}.$$

If $$\mu=\frac{a}{c},$$

then $$T=\frac{aW}{a\sin\theta+a\cos\theta}=\frac{W}{\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)};$$

hence T is least when $\cos\left(\theta-\frac{\pi}{4}\right)$ is greatest, *i.e.* when $\theta=\frac{\pi}{4}$.

41. If A alone slip it must turn round B, so that the friction at A is perpendicular to AB.

Hence, resolving perpendicularly to AB, we have

$$P\sin\theta=\mu W \qquad \text{(1)}.$$

This requires that P should not be $<\mu W$. But since B is not to be on the point of motion the tension of the string must be $<\mu W$, and hence

$$P\cos\theta<\mu W \qquad \text{(2)}.$$

Hence, from (1) and (2), we have, by squaring and adding,

$$P<\sqrt{2}\mu W.$$

Next suppose P to be such that both A and B are on the point of motion so that P is $>\sqrt{2}\mu W$. The friction at B is μW in the direction AB.

Hence the tension T of the string equals μW.

Since the friction μW at A, the tension μW, and P are in equilibrium, the first two must be equally inclined to the direction of P.

Hence $$P=2\mu W\cos\theta,$$

so that $$\cos\theta=\frac{P}{2\mu W}.$$

If $P=\sqrt{2}\mu W$, these two cases coincide and both particles are on the point of motion when θ is 45°.

42. If R and S be the normal reactions of the other two beams on AB, at A and C respectively, since R and S support the weight W, they are given by

$$\frac{R}{b-a}=\frac{S}{a}=\frac{W}{b};$$

if P be the force applied at B perpendicular to BA, and the beam moves, it must move both at A and C or at one of those points. If it move at both, then all the friction possible is exerted, but less if only at one. Thus if the beam begin to move round A, moments about A give

$$P\,.\,2a=\mu S\,.\,b=\mu\,.\,\frac{Wa}{b}\,.\,b,\ i.e.\ P=\frac{\mu W}{2}.$$

If the beam begin to move round C, moments about C give

$$P\,(2a-b)=\mu R\,.\,b=\mu\,.\,\frac{b-a}{b}W\,.\,b,\ i.e.\ \ P=\mu W\frac{b-a}{2a-b}.$$

Hence on the whole the least value of P must be the lesser of the two $\frac{\mu W}{2}$ and $\mu W\frac{b-a}{2a-b}$.

[The former is the lesser if $2a-b<2b-2a$, *i.e.* if $b>\frac{4}{3}a$.]

43. Let R and S be the normal reactions at C and D and let AC be x. Since R and S support the weight W of the beam, we easily have

$$\frac{R}{b+x-a}=\frac{S}{a-x}=\frac{W}{b}\ \ldots\ldots\ldots\ldots(1).$$

Since both C and D are on the point of motion at the same time, the rod must be on the point of turning about some point in CD; the friction μS at D acts therefore in a direction opposite to P, and the friction μR at C acts in the same direction as P.

For the equilibrium of these parallel forces, we have

$$P+\mu R=\mu S,$$

and $$P\,(2a-b-x)=\mu Rb.$$

By (1), these two equations give

$$P=\mu\frac{W}{b}[2a-2x-b]\ldots\ldots\ldots\ldots(2),$$

and $$P = \mu W \frac{b + x - a}{2a - b - x} \qquad (3).$$

Equating these two values of P, we have easily the equation

$$2x^2 - 2x(3a - b) + 4a^2 - 3ab = 0 \qquad (4).$$

If b be not greater than $\frac{4a}{3}$, *i.e.* if $(4a - 3b)$ be positive, both roots of this equation are real and positive.

The lesser root only is admissible; for the greater value of x would make S negative, which is impossible.

44. Let AB be the beam, A and B being its points of contact with planes inclined at α and β to the horizon respectively. Let the beam be on the point of slipping downwards at A on the steeper plane and upwards at B; then the full amount of friction is exerted at each end. Let R and S be the reactions of the planes, and λ and λ' be the angles of friction respectively. Then taking moments about the centre of the beam, we have

$$S \cos(\beta + \lambda') = R \cos(\alpha - \lambda);$$

also, resolving horizontally, we have

$$S \sin(\beta + \lambda') = R \sin(\alpha - \lambda).$$

Hence, by division,

$$\tan(\beta + \lambda') = \tan(\alpha - \lambda);$$

all the angles are acute, and $\beta + \lambda'$ is acute, so that

$$\beta + \lambda' = \alpha - \lambda,$$

i.e. $$\lambda + \lambda' = \alpha - \beta.$$

Hence the equilibrium will be limiting if the difference of the inclinations be just equal to the sum of the angles of friction.

Otherwise thus: Take the extreme case of equilibrium. The directions of R, S and W (acting at G the centre of the beam) must pass through a point O, and G is vertically below O. Then we have the

$$\angle OAB = \frac{\pi}{2} - \alpha + \lambda, \text{ and the } \angle OBA = \frac{\pi}{2} - \beta - \lambda';$$

and since $AG = GB$, and OG is perpendicular to AB, the $\angle OAB =$ the $\angle OBA$; hence

$$\frac{\pi}{2} - \alpha + \lambda = \frac{\pi}{2} - \beta - \lambda', \quad i.e. \quad \lambda + \lambda' = \alpha - \beta;$$

and there will be equilibrium if $\alpha - \beta$ be not greater than $\lambda + \lambda'$.

45. Let AB be the rod, resting on the horizontal plane at A and on the inclined plane at B. Let R and S be the reactions at A and B respectively, and G be the centre of the rod of weight W. Then, resolving horizontally and taking moments about G, we have (since $\mu = \tan \lambda$)

$$\mu R = S(\sin \alpha - \mu \cos \alpha),$$

i.e. $$R \sin\lambda = S \sin(\alpha-\lambda) \qquad (1)$$

and ($2a$ being the length of the rod),

$$Ra(\cos\theta - \mu\sin\theta) = Sa[\cos(\alpha-\theta) + \mu\sin(\alpha-\theta)],$$

i.e. $$R\cos(\theta+\lambda) = S\cos(\alpha-\theta-\lambda).$$

Hence we have

$$\sin\lambda\cos(\alpha-\lambda-\theta) = \sin(\alpha-\lambda)\cos(\theta+\lambda);$$

$$\therefore \tan\theta = \frac{-\sin\lambda\cos(\alpha-\lambda)+\sin(\alpha-\lambda)\cos\lambda}{\sin(\alpha-\lambda)\sin\lambda+\sin\lambda\sin(\alpha-\lambda)} = \frac{\sin(\alpha-2\lambda)}{2\sin\lambda\sin(\alpha-\lambda)}.$$

Also, resolving vertically, we have

$$R + S(\cos\alpha + \mu\sin\alpha) = W,$$

i.e. $$R\cos\lambda + S\cos(\alpha-\lambda) = W\cos\lambda;$$

$$\therefore \frac{R}{\sin(\alpha-\lambda)} = \frac{S}{\sin\lambda} = \frac{W\cos\lambda}{\sin(\alpha-\lambda)\cos\lambda+\sin\lambda\cos(\alpha-\lambda)}$$

$$= \frac{W\cos\lambda}{\sin\alpha}.$$

$$\therefore R = W\cos\lambda\sin(\alpha-\lambda)\operatorname{cosec}\alpha,$$

and $$S = W\cos\lambda\sin\lambda\operatorname{cosec}\alpha.$$

Otherwise thus: Let AO and BO be the directions of the resultant reactions at A and B; then the vertical through G must pass through O. Also since the normal at A is vertical, the angle AOG is λ; and, since the normal at B makes an angle α with the vertical, the angle BOG is $\alpha-\lambda$.

Hence Theorem (1) of Art. 79 gives

$$2\cot OGB = \cot\lambda - \cot(\alpha-\lambda),$$

i.e. $$2\tan\theta = \frac{\sin(\alpha-2\lambda)}{\sin\lambda\sin(\alpha-\lambda)}.$$

Also, if R' and S' be the resultant reactions at A and B, then Lami's Theorem gives

$$\frac{R'}{\sin(\alpha-\lambda)} = \frac{S'}{\sin\lambda} = \frac{W}{\sin\alpha};$$

also the normal reaction at A

$$= R'\cos\lambda = \frac{W\sin(\alpha-\lambda)}{\sin\alpha}\cos\lambda;$$

and the normal reaction at B

$$= S'\cos\lambda = \frac{W\sin\lambda\cos\lambda}{\sin\alpha}.$$

46. Let R be the pressure of the cylinder on one leg, S be the force at the joint (which must be horizontal, by symmetry) and k be the required couple. Then resolving vertically, for the leg, we have

$$R \sin \alpha = W;$$

and taking moments about the joint, we have

$$k = R \cdot c \cot \alpha - W \cdot a \sin \alpha$$
$$= W(c \cot \alpha \operatorname{cosec} \alpha - a \sin \alpha).$$

47. Let A and B be the handles, C be the corner nearest to B, and D be the back corner nearest to A. Then the effect of pulling A will be to turn the drawer slightly round its centre of gravity, so that there will be reactions upon it at C and D, but not at the other corners; and if the limiting coefficient of friction be μ, and Q and R be the normal reactions at D and C respectively, then resolving parallel to the length a and the breadth b of the drawer, we have

$$P \text{ (the pull)} = \mu Q + \mu R,$$

and $$Q = R;$$

$$\therefore P = 2\mu R.$$

Hence $$\mu R \frac{b+c}{2} = \mu Q \frac{b-c}{2} + Q \cdot a,$$

i.e. $$a = \mu c.$$

If this condition be satisfied, any force, however small, would be just on the point of moving the drawer.

If $\mu < \frac{a}{c}$, any force however small will move it.

If $\mu > \frac{a}{c}$, *i.e.* if $a < \mu c$, no force will move it.

48. If the left-hand cord break, the window will tilt slightly so as to press against the frame at A and B, the top left-hand and lower right-hand corners respectively; and if R and S be the normal reactions at A and B respectively, and W be the weight of the window, the tension of the remaining cord $= \frac{W}{2}$; also, if μ be the least coefficient of friction, the window will be on the point of sliding down both at A and B, and hence, for equilibrium, $R = S$, and

$$\mu R + \mu S + \frac{W}{2} = W; \text{ hence } 2\mu R = \frac{W}{2}, \text{ i.e. } W = 4\mu R.$$

Also, taking moments about B, we have, if a and h be the width and the height of the sash respectively,

$$W \cdot \frac{a}{2} = \mu R \cdot a + R \cdot h;$$

$$4\mu \frac{a}{2} = \mu a + h, \quad \text{whence} \quad \mu = \frac{h}{a}.$$

49. If C be the centre of the hoop, O be the bar, and M be the man's hand, OM must be vertical, and clearly OM is least when CO makes the greatest possible angle with the vertical, which is λ; and since $\tan\lambda = 1 \div \sqrt{3}$, $\lambda = 30^\circ$. Then the $\angle COM =$ the $\angle CMO = \lambda = 30^\circ$, and hence the $\angle OCM = 120^\circ$. Hence if CD be drawn perpendicular to OM, we have

$$OM = 2OD = 2\sin 60^\circ = \sqrt{3} \text{ feet.}$$

50. If θ be the required inclination, and R and S be the normal reactions of the pegs, then resolving horizontally, we have

$$R\cos\theta = S\sin\theta.$$

Also, taking moments about the centre of the square, we have

$$R(c\sin\theta - a) = S(c\cos\theta - a);$$

$$\therefore \sin\theta(c\sin\theta - a) = \cos\theta(c\cos\theta - a),$$

$$\therefore c(\sin^2\theta - \cos^2\theta) = a(\sin\theta - \cos\theta);$$

hence either $\sin\theta = \cos\theta$, and then $\theta = 45^\circ$;

or $c(\sin\theta + \cos\theta) = a$, so that $c^2(1 + \sin 2\theta) = a^2$,

and then $\sin 2\theta = \dfrac{a^2 - c^2}{c^2}$, *i.e.* $\theta = \dfrac{1}{2}\sin^{-1}\dfrac{a^2 - c^2}{c^2}$.

Otherwise thus: One position of equilibrium, viz. when one of the edges of the square is inclined at an angle of 45° to the horizon, is obvious. For another position, let $ABCD$ be the square (A being its lowest corner), G be its centre, and W be its weight. Let R and S be the reactions at the pegs P and Q respectively. The directions of R, S and W must meet in a point O, and G is vertically above O. Let the vertical through G meet the line PQ in V. Then $POQA$ is a rectangle, so that $AO = PQ = c$. We have

$$\frac{AG}{AO} = \frac{\sin AOG}{\sin AGO} = \frac{\sin AOV}{\cos GEV}$$

(E being the point where AG meets PQ)

$$= \frac{\sin(90^\circ - 2\theta)}{\cos(45^\circ + \theta)};$$

$$\therefore \frac{a\sqrt{2}}{c} = \frac{\cos 2\theta}{\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)} = \frac{\sqrt{2}(\cos^2\theta - \sin^2\theta)}{\cos\theta - \sin\theta} = \sqrt{2}(\cos\theta + \sin\theta).$$

$$\therefore \frac{a^2}{c^2} = 1 + \sin 2\theta.$$

$$\therefore \sin 2\theta = \frac{a^2 - c^2}{c^2}, \quad i.e. \quad \theta = \frac{1}{2}\sin^{-1}\frac{a^2 - c^2}{c^2}.$$

51. Let S be the mutual normal action between A and B (or C), and R be the normal reaction of the wall on B (or C). For the equilibrium of B, resolving parallel and perpendicular to the wall, we have

$$\mu R = S\cos 60^\circ - \mu S \cos 30^\circ,$$

and
$$R = S\sin 60^\circ + \mu S \sin 30);$$

i.e.
$$\mu R = \frac{S}{2}(1 - \mu\sqrt{3}), \text{ and } R = \frac{S}{2}(\sqrt{3} + \mu);$$

hence, by division,
$$\mu = \frac{1 - \mu\sqrt{3}}{\sqrt{3} + \mu},$$

whence
$$\mu^2 + 2\mu\sqrt{3} + 3 = 4,$$

i.e.
$$\mu + \sqrt{3} = 2, \text{ and } \mu = 2 - \sqrt{3};$$

hence if the coefficient of friction be equal to, or greater than, $2 - \sqrt{3}$ no motion will ensue.

52. Let C represent the ideal axis, ADB the axle on which the machine turns, b be the radius of ADB, and G be the centre of gravity of the machine. Draw a vertical line through G to meet the axle in a point D above G. The reaction of the axle must therefore act in a vertical line through D. Now ϕ is the greatest angle that the resultant reaction at D can make with CD. Hence the greatest value of CDG is ϕ.

But
$$\frac{\sin CGD}{\sin CDG} = \frac{CD}{CG} = \frac{b}{a}.$$

Hence
$$\sin\theta = \sin CGD = \frac{b}{a}\sin\phi$$

gives the greatest value of θ.

53. Let R and S be the normal reactions between the plane and the particle, and the plane and the table respectively, and μR and F the corresponding frictional actions. For the particle, resolving perpendicular to the plane, we have

$$R = w\cos\alpha \quad \text{.............................(1)}.$$

Also, for the plane, resolving vertically and horizontally, we have

$$S + \mu R \sin\alpha = R\cos\alpha + W,$$

and
$$F = \mu R\cos\alpha + R\sin\alpha;$$

hence, by (1), we have

$$S = W + (\cos\alpha - \mu\sin\alpha)\, w\cos\alpha,$$

and
$$F = (\mu\cos\alpha + \sin\alpha)\, w\cos\alpha;$$

hence the particle will move up the plane before the plane slides on the table if $F < \mu S$, *i.e.* if

$$(\mu\cos\alpha + \sin\alpha)\, w\cos\alpha < \mu W + \mu w\cos\alpha\,(\cos\alpha - \mu\sin\alpha),$$

i.e. if
$$\mu W > (1 + \mu^2)\, w\cos\alpha\sin\alpha.$$

54. The vertical through B must pass through the centre of gravity of W' and W, and the resultant reaction at B

$$=W+W'.$$

Let C be the centre of the section, and CA meet the vertical through B in E. Then we have

$$\frac{CE}{EA}=\frac{W}{W'},$$

so that
$$\frac{CE}{CA}=\frac{W}{W+W'}=\frac{CE}{CB}.$$

$$\therefore \frac{\sin\alpha}{\sin(\theta+\alpha)}=\frac{W}{W+W'},$$

i.e.
$$W\sin(\theta+\alpha)=(W+W')\sin\alpha.$$

Otherwise thus: Moments about C give

$$(W+W')\,BC\sin\alpha=W\,.\,CA\cos\left[\theta-\left(\frac{\pi}{2}-\alpha\right)\right],$$

i.e.
$$(W+W')\sin\alpha=W\cos\left[\frac{\pi}{2}-(\theta+\alpha)\right]=W\sin(\theta+\alpha).$$

55. Let Q be the mutual normal action of the spheres; R and S be the normal reactions on them of the bowl respectively, and $W_1>W_2$. Then we have

$$W_1=R\cos\alpha+\mu R\sin\alpha+\mu Q \quad\text{.............(1)},$$

and
$$W_2=S\cos\alpha-\mu S\sin\alpha-\mu Q \quad\text{(2)}.$$

Also, taking moments about the centre of each sphere, we have

$$\mu R=\mu Q,\text{ and }\mu Q=\mu S,\ i.e.\ R=Q=S.$$

Hence (1) and (2) become

$$W_1=[\cos\alpha+\mu(1+\sin\alpha)]R,$$

and
$$W_2=[\cos\alpha-\mu(1+\sin\alpha)]R;$$

hence, by division,

$$\frac{\cos\alpha+\mu(1+\sin\alpha)}{\cos\alpha-\mu(1+\sin\alpha)}=\frac{W_1}{W_2}.$$

Therefore
$$\frac{\mu(1+\sin\alpha)}{\cos\alpha}=\frac{W_1-W_2}{W_1+W_2},$$

$$\mu=\frac{W_1-W_2}{W_1+W_2}\cdot\frac{\cos\alpha}{1+\sin\alpha};$$

also
$$\frac{\cos\alpha}{1+\sin\alpha}=\frac{\cos\alpha(1-\sin\alpha)}{1-\sin^2\alpha}=\frac{1-\sin\alpha}{\cos\alpha}$$

$$=\sec\alpha-\tan\alpha=\tan\left(45^\circ-\frac{\alpha}{2}\right);$$

$$\mu=\frac{W_1-W_2}{W_1+W_2}\tan\left(45^\circ-\frac{\alpha}{2}\right).$$

56. Let A and B be the pegs, the angle ACB be the right angle, and the angle CAB be θ when the point of contact of CB is on the point of moving in the direction BC. Then, if P and Q be the normal reactions at A and B respectively, moments about C give

$$P.AC=Q.BC,$$

i.e. $$\frac{P}{Q}=\frac{BC}{AC}=\tan\theta=\frac{\sin\theta}{\cos\theta}.$$

Also, resolving horizontally, we have

$$\mu P\cos\theta+P\sin\theta=Q\cos\theta-\mu' Q\sin\theta.$$

Hence, eliminating P and Q, we have

$$(\mu\cos\theta+\sin\theta)\sin\theta=(\cos\theta-\mu'\sin\theta)\cos\theta,$$

i.e. $$(\mu+\mu')\sin\theta\cos\theta=\cos 2\theta,$$

i.e. $$\cot 2\theta=\frac{\mu+\mu'}{2}.$$

In the symmetrical position $\theta=\frac{\pi}{4}$; therefore the rods can be turned from that position towards B through an angle $\frac{\pi}{4}-\theta$, *i.e.* $\frac{1}{2}\left(\frac{\pi}{2}-2\theta\right)$,

i.e. $$\frac{1}{2}\left(\frac{\pi}{2}-\cot^{-1}\frac{\mu+\mu'}{2}\right),\ \textit{i.e.}\ \frac{1}{2}\tan^{-1}\frac{\mu+\mu'}{2}.$$

Similarly, from the symmetry of this result as to μ and μ', the system can be turned through an equal angle the other way.

57. The sphere will not slide, since α is less than the angle of friction; if W' be the weight required to keep it from rolling, moments about the point of contact of the sphere with the plane give, a being the radius,

$$W.a\sin\alpha=W'(a\cos\alpha-a\sin\alpha).$$

$$\therefore\ W'=W\frac{}{\cos\alpha-\sin\alpha}.$$

[**Otherwise thus:** Let C be the centre of the sphere, A be its point of contact with the plane, and B be the point from which the required weight W' is suspended; also let CB cut the vertical through A in D. The sphere will not roll if the resultant of W and W' act through A. We have, therefore,

$$CD:DB=W':W,$$

i.e. $$a\tan\alpha:a-a\tan\alpha=W':W,$$

a being the radius, so that $W'(1-\tan\alpha)=W\tan\alpha$,

$$\therefore\ W'=W\frac{\tan\alpha}{1-\tan\alpha}=W\frac{\sin\alpha}{\cos\alpha-\sin\alpha}.\ ]$$

If W' be slightly decreased, the sphere will commence to roll down the plane. If W' be slightly increased, the sphere will commence to roll up the plane.

58. Let O be the centre of the circle, G be the centre of gravity of the rods, $2W$ their weights, A the vertex, $2a$ the length of each rod, r the radius of the circle, and let R and S be the normal reactions at D and E, the points of contact of the upper and lower rods respectively with the circle. Then, resolving vertically, we have

$$(R-S)\cos\alpha+(\mu R+\mu S)\sin\alpha=2W \qquad \text{.........(1)},$$

and resolving horizontally, we have

$$\mu(R-S)\cos\alpha=(R+S)\sin\alpha \qquad \text{.........(2)}.$$

Also, moments about O give

$$\mu R.r+\mu S.r=2W.r,$$

i.e. $$\mu(R+S)=2W \qquad \text{.........(3)}.$$

Hence, by (1), we have

$$(R-S)\cos\alpha=2W-2W\sin\alpha,$$

and, by (2), $$\mu.2W(1-\sin\alpha)=\frac{2W}{\mu}.\sin\alpha;$$

$$\therefore\ \mu^2-\mu^2\sin\alpha=\sin\alpha \qquad \text{.........(4)}.$$

$$\therefore\ \sin\alpha=\frac{\mu^2}{1+\mu^2}=\frac{\tan^2\epsilon}{1+\tan^2\epsilon}=\sin^2\epsilon.$$

$$\therefore\ \sin\epsilon=\sqrt{\sin\alpha}.$$

Again, we have $$AG+GO=AO,$$

i.e. $$a\cos\alpha+r=r\operatorname{cosec}\alpha.$$

$$\therefore\ r=\frac{a\sin\alpha\cos\alpha}{1-\sin\alpha};$$

and the joining string will not meet the circle if $AG>2r$,

i.e. if $$a\cos\alpha>\frac{2a\sin\alpha\cos\alpha}{1-\sin\alpha}$$

i.e. if $$1-\sin\alpha>2\sin\alpha,\ \textit{i.e.}\ \text{if}\ \sin\alpha<\frac{1}{3}.$$

If then T be the tension of the string, taking moments about A for the equilibrium of the upper rod, we have

$$T.2a\cos\alpha+W.a\cos\alpha=R(a+a\sin\alpha),$$

i.e. $$2T\cos\alpha+W\cos\alpha=R(1+\sin\alpha);$$

and for the equilibrium of the lower rod, we have

$$2T\cos\alpha-W\cos\alpha=S(1+\sin\alpha);$$

hence, by addition,

$$4T\cos\alpha=(R+S)(1+\sin\alpha);$$

but, from (3) and (4),

$$R+S=\frac{2W}{\mu}=2W\sqrt{\frac{1-\sin\alpha}{\sin\alpha}};$$

hence $$4T\sqrt{1-\sin^2 a}=2W$$

$$\therefore\ 2T=W\sqrt{\frac{1+\sin a}{\sin a}}=W\sqrt{1+\operatorname{cosec} a},\ \textit{i.e.}\ T=\frac{W}{2}\sqrt{1+\operatorname{cosec} a}.$$

Otherwise thus: Construct the figure as before.

Let the directions of R, S and W meet in the point K; then K is vertically below G. We have

$$\text{the } \angle DKG=\epsilon-a,$$

and $$\text{the } \angle EKG=\epsilon+a,$$

so that $$\text{the } \angle DKE=(\epsilon+a)-(\epsilon-a)=2a;$$

hence a circle will go round $DAKEO$, diameter AO. Also,

$$\text{the } \angle OKG=\epsilon-a+a=\epsilon,\ DO=AO\sin a,$$

and $$OK=AO\sin\epsilon;$$

but $$DO=DG=OK\sin\epsilon.$$

$$\therefore\ \sin^2\epsilon=\sin a,\ \text{and}\ \sin\epsilon=\sqrt{\sin a}.$$

As before, the joining string will not meet the circle if

$$\sin a<\frac{1}{3}.$$

Again, for the equilibrium of the upper rod, taking moments about A, we have $$W.a\sin a+T.2a\cos a=R\cos\epsilon.r\cot a,$$

i.e. $$(W+2T)\,a=\frac{Rr\cos\epsilon}{\sin a};$$

and for the equilibrium of the lower rod, we have

$$(W-2T)\,a=-\frac{Sr\cos\epsilon}{\sin a};\ \text{hence, by division,}\ \frac{W+2T}{W-2T}=-\frac{R}{S}.$$

By Lami's Theorem, we have

$$\frac{R}{S}=\frac{\sin(\epsilon+a)}{\sin(\epsilon-a)};$$

$$\therefore\ \frac{2T+W}{2T-W}=\frac{\sin(\epsilon+a)}{\sin(\epsilon-a)}.$$

$$\therefore\ \frac{2T}{W}=\frac{\sin(\epsilon+a)+\sin(\epsilon-a)}{\sin(\epsilon+a)-\sin(\epsilon-a)}=\frac{2\sin\epsilon\cos a}{2\cos\epsilon\sin a}=\frac{\sin\epsilon\cos a}{\sqrt{1-\sin^2\epsilon}.\sin a}.$$

$$\therefore\ T=\frac{W}{2}.\frac{\sqrt{\sin a}.\cos a}{\sqrt{1-\sin a}.\sin a}$$

$$=\frac{W}{2}.\frac{\sqrt{1-\sin^2 a}}{\sqrt{\sin a(1-\sin a)}}=$$

$$=\frac{W}{2}\sqrt{1+\operatorname{cosec} a}.$$

59. Let i be the slope of the plane, P be the required force, and R be the normal reaction of the plane on the beam. Then, resolving vertically for the beam, and horizontally for the wedge, we have $W=R\cos i$, and $P=R\sin i$, so that $P=W\tan i$.

Again, if there be friction between the floor and the plane, if S be their mutual normal action, and W' be the weight of the wedge, then, resolving vertically, we have

$$W'+R\cos i=S, \text{ i.e. } S=W'+W.$$

Also, resolving horizontally, we have

$$R\sin i=\mu S, \text{ i.e. } W\tan i=\mu S.$$

Hence we have

$$\mu=\frac{W\tan i}{S}=\frac{W}{W+W'}\tan i.$$

60. Let A, B, C be the points of the disc to which the strings are attached; AD the vertical string through A attached to a fixed point D. When the disc is turned horizontally through θ, let A go to A', and draw $A'N$ perpendicular to DA; then N is a point on the new position of the circle vertically above A, so that $A'N=2a\sin\frac{\theta}{2}$. Let ϕ be the inclination of $A'D$ to the vertical, so that

$$\sin\phi=\frac{A'N}{A'D}=\frac{2a\sin\frac{\theta}{2}}{b}.$$

Let T be the tension of each of the three strings so that, by resolving vertically, $3T\cos\phi=W$. The horizontal component of T along $A'N=T\sin\phi$, and its moment about the centre of the disc

$$=T\sin\phi\,.\,a\cos\frac{\theta}{2}=W\,.\,\frac{a}{3}\tan\phi\cos\frac{\theta}{2}.$$

Therefore required couple $=W\,.\,a\tan\phi\cos\frac{\theta}{2}=$ given answer, on substitution for ϕ.

61. Let A be the junction of the rods, B and C be the highest positions of the rings, B' and C' be the lowest positions, and T and T' be the tensions of the string respectively. Consider one ring, at B and B'. By Lami's Theorem we have, in the highest position,

$$\frac{T}{W}=\frac{\sin\left(\frac{\pi}{2}+\alpha+\beta\right)}{\sin(\alpha+\beta)}=\cot(\alpha+\beta);$$

and in the lowest position, $\frac{T'}{W}=\cot(\alpha-\beta)$.

Now $\quad T=\lambda\,.\,\frac{AB\sin\alpha-a}{a}$, and $T'=\lambda\,.\,\frac{AB'\sin\alpha-a}{a}$;

$$\therefore\ T'-T=\frac{\lambda(AB'-AB)\sin\alpha}{a};$$

$$\therefore\ BB'=\lambda^{-1}a\operatorname{cosec}\alpha\,(T'-T)=W\lambda^{-1}a\operatorname{cosec}\alpha\,[\cot(\alpha-\beta)-\cot(\alpha+\beta)].$$

62. Resolving along the slant face of the wedge, we have $W=T$ the tension of the string $=20 \sin 60^\circ = 10\sqrt{3}$.

Let R be the normal reaction of the wedge on the weight. Then (1) when the ring is not attached to the wedge, the horizontal force

$$=R \cos 30^\circ\text{; but } R=20 \cos 60^\circ = 10,$$

$$\text{so that the force} = 10 \cos 30^\circ = 5\sqrt{3} \text{ lbs. wt.}$$

(2) when the ring is attached to the wedge, we have the force

$$=5\sqrt{3} - T \cos 60^\circ = 5\sqrt{3} - 5\sqrt{3} = 0.$$

Again, when the slant face of the edge is rough, we have an additional force μR along the wedges and therefore

$$W+\mu R = 20 \sin 60^\circ,$$

and $$R=20 \cos 60^\circ;$$

$$\therefore\ W = 10\sqrt{3} - 10\mu = 10\sqrt{3} - 10 \cdot \frac{1}{\sqrt{3}} = \frac{20\sqrt{3}}{3}.$$

Also, for case (1), the horizontal force

$$=R \cos 30^\circ - \mu R \cos 60^\circ$$

$$=10 \times \frac{\sqrt{3}}{2} - \frac{10}{\sqrt{3}} \times \frac{1}{2} = 5\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) = \frac{10}{\sqrt{3}} \text{ lbs. wt.};$$

and for case (2), the force

$$=\frac{10}{\sqrt{3}} - W \cos 60^\circ = \frac{10}{\sqrt{3}} - \frac{10}{\sqrt{3}} = 0.$$

63. Let BCA be the crowbar, C being its point of contact with the cylinder, and B being the end at which the force P, applied at right angles to BA, would keep equilibrium. Let R be the action between the crowbar and the cylinder at C. Then, for the crowbar, taking moments about A (to avoid the action there), we have

$$P \cdot l = R \cdot AC = R \cdot r \tan \frac{\alpha+\beta}{2} \quad \text{.....................(1).}$$

Also, for the cylinder, taking moments about D, the point of contact of the cylinder with the plane (to avoid the action there), we have

$$R \cdot r \sin(\alpha+\beta) = W \cdot r \sin \alpha \text{.....................(2).}$$

Eliminating R between (1) and (2), we have

$$P = \frac{Wr \sin \alpha \tan \dfrac{\alpha+\beta}{2}}{l \sin(\alpha+\beta)} = \frac{Wr}{l} \cdot \frac{\sin \alpha}{2 \cos^2 \dfrac{\alpha+\beta}{2}} = \frac{Wr}{l} \cdot \frac{\sin \alpha}{1+\cos(\alpha+\beta)}.$$

64. If W oz. be suspended from A, N be the corresponding division on the scale, G be the centre of gravity of the plate in CO,

where O is the middle point of AB, and the angle $NCO=\theta$, then, taking moments about C, we have

$$W.AC\sin(45^\circ-\theta)=3.CG\sin\theta,$$

$$\therefore\ W.AO.\sqrt{2}\sin(45^\circ-\theta)=3.\frac{2}{3}.CO\sin\theta.$$

$$\therefore\ W(\cos\theta-\sin\theta)=2\sin\theta.$$

$$\therefore\ W(1-\tan\theta)=2\tan\theta,$$

so that
$$\tan\theta=\frac{W}{2+W};$$

also
$$ON=CO\tan\theta=AO.\frac{W}{2+W},$$

and
$$AN=AO-ON=AO\left(1-\frac{W}{2+W}\right)=AO.\frac{2}{2+W}.$$

Let $W=0$, then $AN=AO$, and the zero graduation is at O

" $W=1$, then $AN_1=\frac{2}{3}AO=\frac{1}{3}AB$;

" $W=2$, then $AN_2=\frac{2}{4}AO=\frac{1}{4}AB$;

" $W=3$, then $AN_3=\frac{2}{5}AO=\frac{1}{5}AB$;

and so on; hence the distances of the successive positions of N from A form a diminishing harmonic progression.

65. If B support the ladder at the point C, if G be the centre of the ladder, θ_n the angle it makes with the horizon, and if AC be n feet, then

$$\sin\theta_n=\frac{d}{n};$$

so
$$\sin\theta_{n-1}=\frac{d}{n-1};$$

also the corresponding heights of G above the ground are

$$\frac{l}{2}\sin\theta_n \text{ and } \frac{l}{2}\sin\theta_{n-1}.$$

Hence the work done by B in passing from the nth to the $(n-1)$th foot

$$=W\frac{l}{2}(\sin\theta_{n-1}-\sin\theta_n)$$

$$=\frac{Wl}{2}\left(\frac{d}{n-1}-\frac{d}{n}\right)=\frac{Wld}{2n(n-1)}.$$

Also, the pressure on the ground at $A=W-P$, where P is the force that B exerts; and this changes sign when B passes G; therefore A must press his feet downwards when B has raised more than half the ladder.

66. Let $ABCD$ be the horizontal section of the drawer, O the middle point of the edge AB in which are the handles. Let E, F be the handles, so that

$$EO = OF = \frac{a}{2}.$$

Let F be the handle nearest to B.

Let the drawer be on the point of motion when a force P is applied at F and the drawer has been pushed in a distance x. If points K, L be taken on CB, DA so that $CK = DL = x$, the action S of the side CB acts at K and the action R of the side AD acts at D. [Cf. Ex. 47.]

The frictions μR and μS at D and C act in the directions DA and CB.

Resolving parallel and perpendicular to BC, we have

$$P = \mu R + \mu S \quad \ldots\ldots(1),$$

$$R = S \quad \ldots\ldots(2).$$

Taking moments about F, we have

$$\mu S\left(b - \frac{a}{2}\right) + R \, . \, c = \mu R\left(b + \frac{a}{2}\right) + S\,(c - x) \quad \ldots\ldots(3),$$

where $AB = 2b$, $BC = c$.

From (2) and (3), we have

$$\mu a = x.$$

Motion is thus just possible in the way required when the drawer has been previously pushed in a distance μa.

67. Let R be the normal reaction of the cone at each of the points where it touches a side. Resolving vertically, we have

$$3R \sin \alpha = 3W,$$

i.e.

$$R = \frac{W}{\sin \alpha} \quad \ldots\ldots(1).$$

The horizontal components, $R \cos \alpha$, of these reactions pass through the centre O of the equilateral triangle ABC. By symmetry, the reaction at each angular point is perpendicular to the line joining it to O.

Consider the horizontal forces acting on the side BC. They are $R \cos \alpha$ acting through the middle point D along DO and two forces, each equal to X, acting perpendicular to OB and OC, through B and C. Resolving perpendicular to BC, we have

$$R \cos \alpha = 2X \sin XCB = 2X \sin 60° = X\sqrt{3}.$$

$$\therefore\ X = \frac{R \cos \alpha}{\sqrt{3}} = \frac{W}{\sqrt{3}} \cot \alpha, \text{ by (1)}.$$

68. When motion is just about to ensue let the thread be inclined at an angle θ to the inclined plane. Let R be the normal reaction of the plane, μR the friction, acting downwards, and W the weight. Resolving along and perpendicular to the plane, we have

$$T\cos\theta=\mu R+W\sin\alpha \quad\ldots\ldots\ldots\ldots\ldots(1),$$

$$R+T\sin\theta=W\cos\alpha \quad\ldots\ldots\ldots\ldots\ldots(2).$$

Also, taking moments about the central line of the spindle, we have

$$Tc=\mu Ra \quad\ldots\ldots\ldots\ldots\ldots(3).$$

(1) and (3) give $\qquad T(a\cos\theta-c)=aW\sin\alpha,$

(2) and (3) give $\qquad T(c+\mu a\sin\theta)=\mu aW\cos\alpha.$

$$\therefore\ \mu a\cos\alpha\,(a\cos\theta-c)=a\sin\alpha\,(c+\mu a\sin\theta).$$

$$\therefore\ \mu[a\cos(\theta+\alpha)-c\cos\alpha]=c\sin\alpha.$$

$$\therefore\ \mu=\frac{c\sin\alpha}{a\cos(\theta+\alpha)-c\cos\alpha}.$$

The least value of μ is thus when the denominator is greatest, that is, when $\theta+\alpha$ is zero, that is, when the free part of the thread is horizontal, and then

$$\mu=\frac{c\sin\alpha}{a-c\cos\alpha}.$$

69. Let O be the middle point of BD, so that X is on OA and $OX=\frac{1}{3}OA$; and Y is on OC and $OY=\frac{1}{3}OC$; and hence XY is parallel to AC. Let AC meet BD in O'. Then if G be the required c.G.,

$$\frac{YG}{XG}=\frac{\text{area } ABD}{\text{area } CBD}=\frac{\text{perpendicular from } A \text{ on } BD}{\text{perpendicular from } C \text{ on } BD}=\frac{AO'}{O'C}=\frac{XU}{UY}.$$

$$\therefore\ \frac{YG}{XY}=\frac{XU}{XY},\ \textit{i.e.}\ \ YG=XU. \ \text{ Hence etc.}$$

70. Let α be the angle of the wedge, R the reaction of the floor on it, F the friction, S the force exerted by the door on the inclined face; this must be normal to the face since the latter is smooth.

Resolving horizontally and vertically, we have

$$S \sin \alpha = F,$$

and
$$S \cos \alpha = R.$$

$$\therefore \tan \alpha = \frac{F}{R}.$$

Now the greatest value of $\frac{F}{R}$ = the coefficient of friction μ, so that there will always be equilibrium provided that $\tan \alpha < \mu$, *i.e.* if $\alpha < \tan^{-1} \mu$, *i.e.* less than the angle of friction.

71. Let O be the centre of the cylinder, A its highest point; when the man has walked to the furthest possible point P let C be the point of contact and G the position of the centre of the plank, so that $CG = \text{arc } CA = r \, . \, \theta$, where $\theta = \angle AOC$ = inclination of the plank to the horizon.

Let R be the normal reaction at C and μR the friction.

Since the plank is just on the point of slipping,

$$\therefore R = \left(W + \frac{W}{n}\right) \cos \theta,$$

and
$$\mu R = \left(W + \frac{W}{n}\right) \sin \theta.$$

$$\therefore \tan \theta = \mu, \text{ so that } \theta = \epsilon.$$

Again, taking moments about C, we have

$$CP \, . \, \frac{W}{n} = CG \, . \, W = r\epsilon W.$$

$$\therefore CP = nr\epsilon, \text{ and } GP = (n+1) r\epsilon.$$

72. If T be the tension of the string, R the reaction of the plane, and μR the friction acting up the plane, then resolving along and perpendicular to the plane

$$T \cos \theta + \mu R = W \sin \alpha \quad \text{.......................(1)},$$

and
$$R - T \sin \theta = W \cos \alpha \quad \text{.......................(2)}.$$

Also, taking moments about the centre, we have

$$T = \mu R \quad \text{..................................(3)}.$$

By (3), we have from (1) and (2),

$$\mu R (\cos \theta + 1) = W \sin \alpha,$$

and
$$R (1 - \mu \sin \theta) = W \cos \alpha.$$

$$\therefore \cos \alpha \mu (\cos \theta + 1) = \sin \alpha (1 - \mu \sin \theta).$$

$$\therefore \mu \cos (\theta - \alpha) = \sin \alpha - \mu \cos \alpha.$$

$$\therefore \sin \lambda \cos (\theta - \alpha) = \sin (\alpha - \lambda).$$

Hence $$\theta = \alpha + \cos^{-1}\left[\frac{\sin(\alpha - \lambda)}{\sin\lambda}\right].$$

If θ be > this angle, the equilibrium will be broken; for it is easily shown that the ratio of the tangential to the normal reaction is thereby increased.

73. Let O be the centre of the sphere, and A the point of contact with the wall. Draw AL making an angle ϵ with AO (in a direction above AO) to meet the vertical through O in L. Then AL is the direction of the resultant reaction through A and hence the required force passes through L. Take OL to represent W in magnitude and draw OK perpendicular to AL. Then by the triangle of forces OK represents a force which would be in equilibrium with the weight LO and a force along AL, and it is the least such force, and

$$= OL \sin OLK = W \cos \epsilon.$$

Hence a force $W \cos \epsilon$ through L parallel to OK, *if it meet the sphere at all*, will be the least force required. It will meet the sphere if the perpendicular on it from $O <$ radius of sphere,

i.e. if $\quad OL \cos OLK < a$, *i.e.* if $a > a \tan \epsilon . \sin \epsilon$,

i.e. if $\quad \sin^2 \epsilon < \cos \epsilon$, *i.e.* if $\cos^2 \epsilon + \cos \epsilon > 1$,

i.e. if $\quad \cos \epsilon > \frac{\sqrt{5} - 1}{2}$, *i.e.* if $\epsilon < \cos^{-1} \frac{\sqrt{5} - 1}{2}$.

If ϵ be greater than this angle the line through L, parallel to KO, does not meet the sphere and it is easily seen that L then lies outside the sphere. Hence the nearest direction to KO of a line through L which does meet it will be that of the tangent LT to the sphere, and this will thus be the direction of the required force X. We then have, if OU be parallel to this tangent to meet AL in U, and

$$\angle UOL = \angle OLT = \theta,$$

$$\sin \theta = \frac{OT}{OL} = \frac{OA}{OL} = \cot \epsilon.$$

$$\therefore \frac{X}{W} = \frac{OU}{OL} = \frac{\sin(90^\circ - \epsilon)}{\sin(90^\circ - \epsilon + \theta)} = \frac{\cos \epsilon}{\cos(\epsilon - \theta)}$$

$$= \frac{1}{\cos \theta + \tan \epsilon \sin \theta} = \frac{1}{\sqrt{1 - \cot^2 \epsilon} + 1}$$

$$= \frac{1 - \sqrt{1 - \cot^2 \epsilon}}{\cot^2 \epsilon} = \tan \epsilon \left[\tan \epsilon - \sqrt{\tan^2 \epsilon - 1}\right].$$

For the limiting value $\epsilon = \cos^{-1} \frac{\sqrt{5} - 1}{2}$, the two values above found will of course coincide.

74. Let O be the hinge, OA the rod, OB the perpendicular from O upon the wall, BP the vertical through P, so that $\angle OAB = a$. Let θ be the angle ABP so that, when the equilibrium is limiting, 2θ is the angle required. The normal reaction R acts through A parallel to BO, and the friction μR acts tangentially to a circle with B as centre, *i.e.* along a line in the wall through A perpendicular to BA.

Taking moments about a line through O parallel to BP, so that the weight and the reaction at O do not enter into the equation of moments, we have

$$R \times AB \sin\theta = \mu R \cos\theta \,.\, OB + \mu R \sin\theta \times O,$$

i.e.
$$R \times OA \cos a \sin\theta = \mu R \cos\theta \,.\, OA \sin a,$$

i.e.
$$\tan\theta = \mu \tan a \qquad\qquad (1).$$

Also, taking moments about OB, we have

$$W \times \frac{AB}{2} \sin\theta = \mu R \times BA,$$

i.e.
$$R = \frac{W}{2\mu} \sin\theta = \frac{W}{2\mu} \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \frac{W}{2} \frac{\tan a}{\sqrt{1+\mu^2 \tan^2 a}} = \frac{W}{2} \cdot \frac{1}{\sqrt{\mu^2+\cot^2 a}}, \text{ by equation (1).}$$

75. Let P and Q be two consecutive angular points of the cube which are on the surface of the hemisphere. Let the plane which bisects PQ at right angles meet the plane base in AOB, O being the centre. Let perpendiculars from P, Q on the plane base meet it in R, S and let RS meet OA in N. Then if a be the radius of the sphere and x the side of the cube,

$$a^2 = OP^2 = PR^2 + OR^2 = PR^2 + RN^2 + ON^2$$

$$= x^2 + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = \frac{3}{2} x^2,$$

so that
$$x = \frac{1}{3} a\sqrt{6}.$$

Let K be the c.g. of the rest of the cube; then K will lie on the radius OC perpendicular to the plane base, and

$$OK \times \left[\frac{2}{3}\pi a^3 - x^3\right] + \frac{x}{2} \times x^3 = \frac{2}{3}\pi a^3 \times \frac{3a}{8} = \frac{1}{4}\pi a^4.$$

$$\therefore\ OK = \frac{\frac{1}{4}\pi a^4 - \frac{x^4}{2}}{\frac{2}{3}\pi a^3 - x^3} = \frac{\frac{1}{4}\pi a^4 - \frac{2a^4}{9}}{\frac{2}{3}\pi a^3 - \frac{2a^3}{9}\sqrt{6}} = \frac{a}{8} \cdot \frac{9\pi - 8}{3\pi - \sqrt{6}}.$$

Let the body be now placed with a point L of the spherical surface in contact with the plane; then the limiting position of the body will be that in which K is vertically over L, and then, since $\angle OLK=a$, we have

$$\frac{\sin LKC}{\sin a}=\frac{OL}{OK}=\frac{a}{x}=\frac{8(3\pi-\sqrt{6})}{9\pi-8},$$

and $$\therefore \sin LKC=\frac{8(3\pi-\sqrt{6})}{9\pi-8}\sin a.$$

Also $\angle LKC=\angle OC$ makes with the vertical

$=\angle$ the plane base makes with the horizon.

76. Let the length l be divided into n equal parts where n is very large and ultimately will be made infinite.

When a length $\frac{pl}{n}$ has been extracted, a length $l-\frac{pl}{n}$ is left in, and the pressure is

$$2\pi r\left(l-\frac{pl}{n}\right)P,$$

and therefore the friction is

$$2\pi\mu rlP\left(1-\frac{p}{n}\right).$$

The work done in extracting the cork a further distance $\frac{l}{n}$ is

$$2\pi\mu rl^2P\left(1-\frac{p}{n}\right)\frac{1}{n}.$$

Hence the total work required is the limit, when n is infinite, of

$$\sum_{p=1}^{p=n}2\pi\mu r^2lP\left(\frac{1}{n}-\frac{p}{n^2}\right)$$

$$=2\pi\mu r^2lP\times\underset{n=\infty}{\text{Lt}}\sum_{p=1}^{p=n}\left(\frac{1}{n}-\frac{p}{n^2}\right)$$

$$=2\pi\mu r^2lP\times\underset{n=\infty}{\text{Lt}}\left[\frac{n}{n}-\frac{\frac{1}{2}n(n+1)}{n^2}\right]$$

$$=2\pi\mu r^2lP\times\underset{n=\infty}{\text{Lt}}\left[1-\frac{1}{2}-\frac{1}{2n}\right]$$

$$=2\pi\mu r^2lP\times\frac{1}{2}=\pi\mu r^2lP.$$

ELEMENTS OF DYNAMICS.

Page 4. Art. 5.

Ex. 1. The required speed

$=[2\pi \times 93000000] \div [365 \times 24 \times 60 \times 60]$ miles per second

$=3875\pi \div 657=$ about 18·5 miles per second.

Ex. 2. The distance from the sun to the earth being 93000000 miles (as given in Ex. 1), the required speed

$=93000000 \div [8 \times 60]$ miles per second

$=193750$ miles per second.

4. Art. 6.

Ex. 1. If the man walk 3 miles east from A to B, and then 4 miles north from B to C, his displacement

$$=AC=\sqrt{3^2+4^2} \text{ miles}=5 \text{ miles},$$

at an angle north of east equal to the $\angle BAC$,

$$i.e. \tan^{-1}\frac{BC}{BA}, \; i.e. \tan^{-1}\frac{4}{3}.$$

Ex. 2. Let OA and AB be the two successive displacements, draw BN perpendicular to OA produced, so that AN and NB are each equal to 1 mile.

The displacement

$$=OB=\sqrt{ON^2+NB^2}=\sqrt{2^2+1^2}=\sqrt{5} \text{ miles};$$

also $$\tan BON=\frac{NB}{ON}=\frac{1}{2}.$$

Page 10. Art. 14.

Ex. 1. The components are $4\cos 45°$ and $4\sin 45°$, *i.e.* each component is $2\sqrt{2}$ miles per hour.

Ex. 2. The required component $=(10\cos 30°)$ feet per second $=5\sqrt{3}$ feet per second.

Ex. 3. If u be the velocity of the body, the required components respectively are $u\cos 60°$ and $u\sin 60°$,

$$i.e.\ \frac{u}{2} \text{ and } u\frac{\sqrt{3}}{2}.$$

EXAMPLES. I. (Pages 13—16.)

3. $8\frac{2}{11}$ miles per hour

$$=\left(8\frac{2}{11}\times\frac{1760\times 3}{60\times 60}\right) \text{ feet per second} = 12 \text{ feet per second.}$$

The component velocities are therefore 9 feet per second and 12 feet per second, so that the resultant velocity

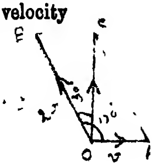
$$=\sqrt{9^2+(12)^2}=15 \text{ feet per second.}$$

Hence the ball passes over 45 feet in 3 seconds. Also its direction in space is inclined at an angle $\tan^{-1}\frac{3}{4}$ to the direction of the ship.

4. The boat is rowed three times as fast as the current flows. Hence the required distance

$$=\frac{1}{3}\times 300 \text{ feet} = 100 \text{ feet.}$$

5. Let v be the velocity of the current, O be the starting point, and C be the point to be reached. Let OA and OB represent in magnitude and direction v and $2v$ respectively. Complete the parallelogram $OACB$; then the diagonal OC represents the resultant velocity in magnitude and direction. We have



$$\sin BOC=\frac{BC}{OB}=\frac{OA}{OB}=\frac{v}{2v}=\frac{1}{2}, \ i.e.\ \angle BOC=30°,$$

and therefore the required inclination $=$ the $\angle AOB = 120°$.

6. Construct as in the last example. Here we have

$$\sin BOC=\frac{4}{6}=\frac{2}{3}.$$

7. Here $$\sin BOC = 1\tfrac{1}{2} \div 2\tfrac{1}{2} = \frac{3}{5},$$

so that $$\angle AOB = 90^\circ + \sin^{-1}\frac{3}{5} = \cos^{-1}\left(-\frac{3}{5}\right) = 126^\circ\, 52'.$$

In order to cross in the shortest time, the swimmer should devote his whole energies to crossing, it being immaterial how far he may drift down; therefore he should swim, relatively to the stream, perpendicular to its direction. Thus the resultant direction of his motion makes with the current an angle

$$\tan^{-1}\frac{2\frac{1}{2}}{1\frac{1}{2}}, \ \textit{i.e.}\ \tan^{-1}\frac{5}{3}, \ \textit{i.e.}\ 59^\circ\, 2'.$$

8. The northward displacement in one hour

$$= (8\sqrt{3}\cos 30^\circ) \text{ miles} = 12 \text{ miles};$$

also the westward displacement

$$= (8\sqrt{3}\sin 30^\circ) \text{ miles} = 4\sqrt{3} \text{ miles}.$$

Hence the required answers are $4\sqrt{3}$ miles per hour, and 12 miles per hour.

9. Let P be the point at which they strike in t hours; then

$$AP = 10t, \text{ and } BP = 10\sqrt{3}t.$$

We have $$\frac{\sin ABP}{\sin 60^\circ} = \frac{AP}{BP} = \frac{1}{\sqrt{3}},$$

so that $$\sin ABP = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{2}, \ \textit{i.e.}\ \angle ABP = 30^\circ;$$

therefore Y starts at an angle of 150° with AB produced. Also the $\angle APB = 90^\circ =$ the angle at which the ships strike. Again,

$$AP = 5\cos 60^\circ, \ \textit{i.e.}\ 10t = \frac{5}{2} \text{ miles},$$

so that $$t = \frac{1}{4} \text{ hour} = 15 \text{ minutes}.$$

10. 8 miles per hour

$$= \left(8 \times \frac{1760 \times 3}{60 \times 60}\right) \text{ feet per second}$$

$$= \frac{176}{15} \text{ feet per second}.$$

Let OA and OB represent in magnitude and direction the velocities $\frac{176}{15}$ feet per second and 16 feet per second respectively. Complete

the parallelogram $OACB$; then the diagonal OC represents the resultant velocity in magnitude and direction. We have

$$\cos AOB = -\sin BOC = -\frac{176}{15\times 16} = -\frac{11}{15},$$

i.e. the required direction is at an angle

$$\cos^{-1}\left(-\frac{11}{15}\right)$$

with the direction of the car's motion.

11. Let OA and OB represent in magnitude and direction the velocity of the ship and the velocity of the current respectively. Complete the parallelogram $OADB$; then the diagonal OD represents the resultant velocity in magnitude and direction; it is equal to $\sqrt{4^2+3^2}$, *i.e.* 5 feet per second; also

$$\angle DOA = \tan^{-1}\frac{AD}{AO} = \tan^{-1}\frac{3}{4},$$

i.e. the direction of OD is an angle $\tan^{-1}\frac{3}{4}$ east of north. Let OC represent the velocity of the sailor. The required velocity is the resultant velocity of OD and OC, and $=\sqrt{5^2+2^2}=\sqrt{29}$ feet per second, at an angle $\tan^{-1}\frac{2}{5}$ with OD.

12. If x and y be the required components, we have

$$\frac{x}{\sin 45^\circ} = \frac{y}{\sin 30^\circ} = \frac{u}{\sin 75^\circ} = \frac{2\sqrt{2}u}{\sqrt{3}+1} = u\sqrt{2}\,(\sqrt{3}-1).$$

Hence $\quad x = u(\sqrt{3}-1),\;$ and $\; y = \frac{u}{2}(\sqrt{6}-\sqrt{2}).$

13. As in Art. 13, since the velocity 13 is equal and opposite to the resultant of the velocities 7 and 8, if θ be the required angle, we have

$$(13)^2 = 7^2 + 8^2 + 2\,.\,7\,.\,8\cos\theta,$$

whence $\quad \cos\theta = \frac{1}{2},\;$ *i.e.* $\;\theta = 60^\circ.$

14. Velocities, each equal to 3, in the given directions destroy one another, so that we have to find the resultant of 16 and 6 at an angle of 120°. Hence

$$V^2 = (16)^2 + 6^2 + 2\,.\,16\,.\,6\cos 120^\circ = 196,\; i.e.\; V = 14.$$

Also $\quad \tan\theta = \frac{6\sin 120^\circ}{16 + 6\cos 120^\circ} = \frac{3\sqrt{3}}{13},$

so that $\quad \cos\theta = \frac{13}{14},\;$ *i.e.* $\;\theta = \cos^{-1}\frac{13}{14}.$

15. Here

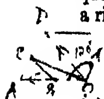
$$V\cos\theta = u + 2u\cos 60^\circ - 3\sqrt{3}u\cos 30^\circ + 4u\cos 60^\circ = -\frac{u}{2},$$

and $$V\sin\theta = 2u\sin 60^\circ + 3\sqrt{3}u\sin 30^\circ - 4u\sin 60^\circ = \frac{\sqrt{3}u}{2}.$$

Hence $$V^2 = \frac{u^2}{4}(1+3) = u^2, \text{ so that } V = u.$$

Also $$\tan\theta = \frac{\sqrt{3}u}{2} \div \left(-\frac{u}{2}\right) = -\sqrt{3}, \text{ i.e. } \theta = 120^\circ.$$

16. Let OA and OB be the original velocities. Complete the parallelogram $OADB$, and let C be the middle point of AD. By the question the angle AOD is bisected by OC, so that the angle OCA is a right angle. Hence

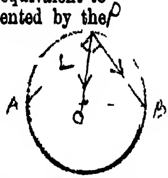


$$\sin AOC = \frac{AC}{OA} = \frac{1}{2},$$

i.e. the $\angle AOC$ is 30°, and therefore the $\angle AOB$ is 120°.

Fig.(16)

17. If AOB be the diameter and P be the point on the circle, then, by Art. 16, Cor. 2, the velocities PA and PB are equivalent to $2PO$, where O is the centre, *i.e.* to a velocity represented by the diameter through P.



EXAMPLES. II. (Pages 21—24.)

Fig.(17)

1. 30 miles per hour

$$= \left(30 \times \frac{1760 \times 3}{60 \times 60}\right) \text{ feet per second} = 44 \text{ feet per second.}$$

Hence the required velocity is the resultant of 33 feet per second at right angles to the train and of 44 feet per second in a direction opposite to that of the train, *i.e.* is $\sqrt{(41)^2+(33)^2}$, *i.e.* 55 feet per second, at an angle $\tan^{-1}\left(-\frac{3}{4}\right)$ with the direction of the train's motion.

2. If u be the required velocity at an angle θ west of north, u is compounded, by the principle of Art. 22, of 16 miles per hour north and 12 miles per hour west, and

$$= \sqrt{(16)^2+(12)^2} = 20 \text{ miles per hour};$$

also $$\theta = \tan^{-1}\frac{12}{16} = \tan^{-1}\frac{3}{4}.$$

3. The apparent velocity (u) of the second vessel, *i.e.* its velocity relative to the first vessel, is compounded of 15 miles per hour south-east, and $15\sqrt{2}$ miles per hour north; hence it is u at some angle θ east of north, where

$$u\cos\theta = 15\sqrt{2} - \frac{15}{\sqrt{2}} = \frac{15}{\sqrt{2}}, \quad \text{and} \quad u\sin\theta = \frac{15}{\sqrt{2}}.$$

Hence $$u^2 = \frac{(15)^2}{2}.2,$$

and hence $$u = 15 \text{ miles per hour.}$$

Also $$\tan\theta = 1, \text{ so that } \theta = 45°,$$

i.e. the apparent velocity is 15 miles per hour north-east.

4. If u be the true velocity of the wind at an angle θ east of south, then this velocity compounded with 10 miles per hour south-west is $10\sqrt{2}$ miles per hour south. Hence $u = 10$ miles per hour, and $\theta = 45°$, *i.e.* the true velocity is 10 miles per hour towards the south-east.

5. The required velocity is compounded of $(30+6)$ miles per hour north, and 15 miles per hour east, and therefore

$$= \sqrt{(36)^2 + (15)^2} = 39 \text{ miles per hour,}$$

in a direction $\cos^{-1}\frac{15}{39}$, *i.e.* $\cos^{-1}\frac{5}{13}$, north of east.

6. Let the velocity of the train be x feet per second. Then we have $\frac{24}{x} = \frac{1}{2}$, whence $x = 48$. The velocity of the train is therefore $\frac{48 \times 60 \times 60}{1760 \times 3}$, *i.e.* $32\frac{8}{11}$, miles per hour.

7. Let u be the speed of the rain, and θ be its direction with the vertical; then (1) u compounded with 2 miles per hour in the opposite direction is vertical, so that $u\sin\theta = 2$; and (2)

$$\frac{u}{\sin 45°} = \frac{4}{\sin(\theta + 45°)},$$

so that $$u(\sin\theta + \cos\theta) = 4 = 2u\sin\theta;$$

$$\therefore \sin\theta = \cos\theta, \text{ i.e. } \theta = 45°;$$

also $$u = 2\operatorname{cosec}\theta = 2\sqrt{2} \text{ miles per hour.}$$

Otherwise thus: Let OB represent the real speed of the rain, and OC be vertical. Draw OA horizontal and opposite to the direction of the man, and let it represent in magnitude the speed, 2 miles an hour, of the man. Complete the parallelogram $OBCA$. Produce OA to A', making $AA' = OA$; then OA' represents in magnitude the increased speed, 4 miles an hour. Complete the parallelogram $OBC'A'$, and join OC'. The $\angle OC'B = 45°$; also $CC' = AA' = 2$, so that $OC = 2$, and $CB = 2$; hence $OB = 2\sqrt{2}$ miles per hour, and the $\angle BOC = 45°$.

8. Let OA be drawn towards the east and be equal and opposite to the velocity of the steamer. Draw OB towards the south-east and equal to the apparent velocity of the wind. Complete the parallelogram $OABC$. Then OC represents the actual velocity of the wind. Also

$$OC = AB = \sqrt{(14)^2 + 7^2 - 2 \, . \, 14 \, . \, 7 \cos 45^\circ}$$
$$= 7\sqrt{4 + 1 - 2\sqrt{2}} = 7\sqrt{5 - 2\sqrt{2}} \text{ miles per hour.}$$

9. If $\sin\theta = \frac{3}{5}$, then $\cos\theta = \frac{4}{5}$. Let v be the velocity of the shot; then its components along and perpendicular to the train are $\frac{4v}{5}$ and $\frac{3v}{5}$ respectively. Hence we have

$$\frac{4v}{5} - 28 : \frac{3v}{5} = 6 : 8,$$

whence $v = 80$ miles per hour. Now 80 miles per hour

$$= \frac{80 \times 1760 \times 3}{60 \times 60} \text{ feet per second} = \frac{352}{3} \text{ feet per second;}$$

hence the required time $= 8 \div \frac{3v}{5} = \frac{40}{352} = \frac{5}{44}$ sec.

10. The relative velocity of each train with regard to the other, since they are moving in opposite directions, is $(20 + 30)$ miles per hour, *i.e.* 50 miles per hour; and to clear each other from the moment when they first meet either has to pass over a relative distance equal to the sum of their lengths, *i.e.* 400 feet. Hence the required time is

$$\frac{400 \times 60 \times 60}{50 \times 1760 \times 3} \text{ seconds, } i.e. \ 5\tfrac{5}{11} \text{ seconds.}$$

11. The length of a steam track is proportional to the velocity of the train relatively to the wind. Hence, if u be the velocity of each train, and v be that of the wind, we have

$$u + v : u - v = 2 : 1, \ i.e. \ u + v = 2(u - v), \text{ whence } u = 3v.$$

12. Let v denote the common speed, 15 miles per hour.

If the first ship was at O at noon, the second ship was then at A, where $OA = \frac{3}{2}v$ miles. Let P and Q be simultaneous positions of the ships at t hours past noon, so that $OP = vt$ and

$$QO = v\left(t - \frac{3}{2}\right).$$

Hence
$$PQ = \sqrt{v^2\left(t - \frac{3}{2}\right)^2 + v^2t^2} = v$$
$$= \frac{v}{\sqrt{2}}\sqrt{4t^2 - 6t + \frac{9}{2}} = \frac{v}{\sqrt{2}}\sqrt{\left(2t - \frac{3}{2}\right)^2 + \frac{9}{4}}.$$

Hence the least value of PQ is when $t=\frac{3}{4}$ hour, *i.e.* the required time is 12.45 p.m., and this value is then

$$\frac{v}{\sqrt{2}}\times\frac{3}{2},\ i.e.\ \frac{3v\sqrt{2}}{4},\ i.e.\ \frac{45\sqrt{2}}{4},\ i.e.\ 15{\cdot}9 \text{ miles.}$$

13. The ships being at O and A, $OA=10$ miles. Let P and Q be simultaneous positions of the ships after t hours, so that $OQ=12t$ and $AP=16t$. Hence

$$PQ=\sqrt{(12t)^2+(10-16t)^2}=\sqrt{100-320t+400t^2}$$
$$=\sqrt{36+(8-20t)^2}.$$

Hence the least value of PQ is when $t=\frac{8}{20}$ hour $=24$ minutes, and this value is then six miles.

14. The velocity of the first is $\frac{5}{3}$ feet per second, and that of the second is $\frac{5}{4}$ feet per second. The relative velocity of the second with respect to the first is therefore the resultant of $\frac{5}{4}$ perpendicular to BA and of $\frac{5}{3}$ along BA, *i.e.* is $\sqrt{\left(\frac{5}{4}\right)^2+\left(\frac{5}{3}\right)^2}$, *i.e.* $\frac{25}{12}$, *i.e.* $2\frac{1}{12}$ feet per second at an angle $\tan^{-1}\left(\frac{5}{4}\div\frac{5}{3}\right)$, *i.e.* $\tan^{-1}\frac{3}{4}$ with BA. Let P and Q be simultaneous positions of the points at the end of t seconds, so that $BP=\frac{5}{3}(3-t)$, and $BQ=\frac{5}{4}t$.

Hence

$$PQ=\sqrt{\frac{25}{9}(3-t)^2+\frac{25}{16}t^2}=\frac{1}{12}\sqrt{(25t-48)^2+(36)^2}.$$

Hence the least value of PQ is when

$$t=\frac{48}{25}=1\tfrac{23}{25} \text{ second,}$$

and this value then $=\frac{36}{12}=3$ feet.

15. Let OA be drawn westward and be equal and opposite to the speed of the ship. Draw OB south-east and equal to the speed of the wind. Complete the parallelogram $OBCA$, so that OC is the direction of the vane. Now, since the apparent direction, OC, of the wind is towards the S.S.W., therefore the $\angle BOC=67\frac{1}{2}°$, and the $\angle OBC=45°$, so that the $\angle BCO=67\frac{1}{2}°$.

Hence $OB=BC$, and therefore the speed of the ship is equal to that of the wind.

16. Draw OA in a direction between south and east, to represent the velocity of the wind; also draw OB westward and equal to 4 and complete the parallelogram $OACB$. By the conditions of the question the direction OC is due south. Produce OB to D, so that DB is equal to 4, and complete the parallelogram $OAED$. By the question the $\angle COE$ is 45°, and hence the $\angle CEO$ is 45°. Hence

$$OC = CE = DB = 4, \text{ and } AC = OB = 4.$$

Therefore $$OA = 4\sqrt{2},$$

and the $\angle AOC$ is 45°; *i.e.* the required velocity is $4\sqrt{2}$ miles per hour, towards the south-east.

17. Draw OA at an angle θ with the south direction towards the east, to represent the velocity of the wind; also draw OB south west to represent the velocity equal and opposite to the velocity of the person. Complete the parallelogram $OADB$, so that OD is due south. Produce OB to C, so that OB equals BC, and complete the parallelogram $OAEC$. Then the $\angle EOD$ is $\alpha\ (=\cot^{-1} 2)$.

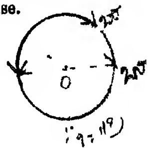
We have $$\frac{AD}{OA} = \frac{\sin\theta}{\sin 45^\circ} = \sqrt{2}\sin\theta,$$

and $$2\frac{AD}{OA} = \frac{EA}{OA} = \frac{\sin(\theta+\alpha)}{\sin(45^\circ-\alpha)} = \frac{\sqrt{2}(\sin\theta\cot\alpha+\cos\theta)}{\cot\alpha - 1}$$
$$= \sqrt{2}\,(2\sin\theta + \cos\theta).$$

Hence $2\sqrt{2}\sin\theta = \sqrt{2}\,(2\sin\theta+\cos\theta)$, so that $\cos\theta = 0$.

Therefore the $\angle DOA$ is a right angle, and the true direction of the wind is towards the east.

18. When the line joining them passes through the centre, their relative velocity has its greatest value $3v$ or its least value v, according as the points are moving in the opposite or in the same sense.



EXAMPLES. III. (Pages 26—28.)

1. The angular velocity

$$= \frac{200 \times 2\pi}{60} \text{ radians per second} = \frac{20\pi}{3} \text{ radians per second.}$$

2. The angular velocity of any point on the wheel about the centre is the angular velocity of the wheel and $= 4 \times 2\pi = 8\pi$ radians per second. Hence the linear velocity of any point on the wheel

$$= 8\pi \times 2 \text{ feet per second} = 16 \times \frac{22}{7} = 50\tfrac{2}{7} \text{ feet per second.}$$

3. The minute hand makes one revolution per hour; hence its angular velocity

$$=\frac{2\pi}{60\times 60}=\frac{\pi}{1800} \text{ radians per second.}$$

Also the velocity of the end

$$=6\times\frac{2\pi}{60\times 60}=\frac{\pi}{300} \text{ feet per second.}$$

4. The respective angular velocities are as

$$\frac{1}{12}\times\frac{1}{60\times 60}:\frac{1}{60\times 60}:\frac{1}{60}, \text{ i.e. as } \frac{1}{12}:1:60;$$

the velocities of the ends are as

$$\frac{\cdot 48}{12}:\cdot 8:\cdot 24\times 60, \text{ i.e. as } 1:20:360.$$

5. The man's rate is equal and opposite to the velocity of the point on which he is standing, and the angular velocity of the wheel

$$=\frac{2\pi}{40} \text{ radians per second;}$$

hence the required rate

$$=\frac{2\pi}{40}\times 20 \text{ feet per second}=\pi \text{ feet per second}$$

$$=3600\pi \text{ feet per hour}=1200\times\frac{22}{7} \text{ yards per hour}$$

$$=\frac{1200}{1760}\times\frac{22}{7} \text{ miles per hour}=2\tfrac{1}{7} \text{ miles per hour.}$$

6. If O be the object, P and Q be any two consecutive positions of the point of observation in the train, and p and q be the corresponding positions of the carriage, then the lines Pp and Qq both pass through O; also during the time that the train moves from P to Q, the carriage moves from p to q. Hence the velocity of the carriage is to that of the train, V, as $pq:PQ$, *i.e.*, from similar triangles, as $D-d:D$; hence the required velocity

$$=\frac{D-d}{D}V.$$

7. Let A be the fixed point on the circumference, O be the centre, AOB be the diameter, and P be any point on the circumference.

Since the point moves with uniform speed, the angle BOP increases at a uniform rate.

Hence the angle BAP, which is half of the angle BOP, must also increase uniformly.

8. Let $ABCD$ be the board, the particle being originally at A. The string has unwrapped itself when it has turned through a right angle about the points D, C and B respectively, *i.e.* when it has described one quarter of the circumference of 3 circles whose radii are a, $2a$, and $3a$, *i.e.* when it has described a distance

$$\frac{\pi}{2} \cdot a + \frac{\pi}{2} \cdot 2a + \frac{\pi}{2} \cdot 3a, \quad i.e. \ 3\pi a.$$

The required time therefore

$$= \frac{3\pi a}{u}.$$

10. 60 miles per hour $= \dfrac{60 \times 1760 \times 3}{60 \times 60} = 88$ feet per second.

The velocity of the centre of the wheel is 88 feet per second, so that the angular velocity of the wheel

$$= \frac{88}{2} = 44 \text{ radians per second.}$$

Taking the figure on p. 27, we have

$$\cos\theta = \frac{1}{2}, \quad i.e. \ \theta = 60^\circ,$$

and hence the tangent at P makes an angle $\frac{\theta}{2}$, *i.e.* 30°, with the horizon. Also the velocity of P

$$= AP \cdot \omega = 4\cos 30^\circ \times 44 = 88\sqrt{3} \text{ feet per second}$$

$$= \frac{88\sqrt{3} \times 60 \times 60}{3 \times 1760} = 60\sqrt{3} \text{ miles per hour.}$$

11. 30 miles per hour = 44 feet per second.

The required angular velocity

$$= 44 \div \frac{3}{2} = \frac{88}{3} \text{ radians per second.}$$

Also the highest point of the wheel moves twice as fast as the centre, so that the velocities of these two points are 88 feet per second and 44 feet per second respectively. Hence the relative velocity

$$= 44 \text{ feet per second} = 30 \text{ miles per hour.}$$

12. Proceed as in the last example.

13. Take the figure on p. 27. The velocity of the highest point

$$B = 2v = 20 \text{ miles per hour.}$$

Let P and P', on opposite sides of BA, be points 3 feet above the ground, and Q and Q', vertically below P and P', be points 1 foot above the ground.

$$\text{The } \angle BOP = \text{the } \angle BOP' = \text{the } \angle AOQ = \text{the } \angle AOQ' = 60^\circ.$$

The velocity of Q or Q' is the resultant of v and v at 120°,

i.e. is $\quad 2 \times 10 \cos 60^\circ$, *i.e.* 10 miles per hour,

at $\pm 60^\circ$ to the horizon. The velocity of P or P' is the resultant of v and v at 60°,

i.e. is $\quad 2 \times 10 \cos 30^\circ$, *i.e.* $10\sqrt{3}$ miles per hour,

at $\pm 30^\circ$ to the horizon.

EXAMPLES. IV. (Page 30.)

2. Let OA represent the original velocity, 3 miles per hour, and OB the final velocity, 4 miles per hour. Complete the parallelogram $AOCB$, with OB diagonal; the required change of velocity is OC which

$$= \sqrt{3^2 + 4^2} = 5 \text{ miles per hour},$$

at an angle $\tan^{-1}\dfrac{4}{3}$ north of west.

3. If OA and OB represent the two velocities of 5 feet per second, then OA equals OB and the triangle OAB is equilateral. Hence, if the parallelogram $OABC$ be completed, $OC = AB$, and the required change of velocity is 5 feet per second, at an angle COA,

i.e. $\quad 60^\circ + \angle OBA$,

i.e. 120° with OA.

4. Let OA represent the original and OB the final velocity, so that OA and OB each represent 20 feet per second, and the $\angle AOB$ is 45°. Hence

$$AB = 2 \times 20 \sin 22\tfrac{1}{2}^\circ = 20\sqrt{2(1 - \cos 45^\circ)} = 20\sqrt{2 - \sqrt{2}};$$

and the $\quad \angle OAB = 90^\circ - 22\frac{1}{2} = 67\frac{1}{2}^\circ$,

so that AB is in a direction N.N.W.

5. The speed $= [2\pi \times 21] \div 11$ feet per second $= 12$ feet per second. The final velocity OB is equal in magnitude to the original velocity OA, and is inclined at 60° to it. Hence AB is 12 feet per second, and the $\angle OAB$ is 60°. Therefore the change is a velocity of 12 feet per second in a direction inclined at 120° to the direction of the original velocity.

EXAMPLES. V. (Pages 39—41.)

1. (1) $v=u+ft=2+3\,.\,5=17$ feet per second;

$$s=ut+\frac{1}{2}ft^2=2\,.\,5+\frac{1}{2}\,.\,3\,.\,25=47\tfrac{1}{2} \text{ feet.}$$

(2) $v=u+ft=7+(-1)\,7=0$;

$$s=ut+\frac{1}{2}ft^2=7^2-\frac{1}{2}\,.\,7^2=24\tfrac{1}{2} \text{ feet.}$$

(3) $v^2=u^2+2fs$, so that $3^2=8^2+18f$,

whence $$f=-3\frac{1}{18} \text{ ft.-sec. units;}$$

$$v=u+ft, \text{ so that } 3=8-\frac{55}{18}t,$$

whence $$t=1\tfrac{7}{11} \text{ second.}$$

(4) $v^2=u^2+2fs$, so that $(-6)^2=u^2+27$,

whence $$u=\pm 3 \text{ feet per sec.;}$$

$$v=u+ft, \text{ so that } -6=\pm 3-\frac{3}{2}t,$$

whence $t=6$ seconds, or 2 seconds.

2. $v=ft=2\,.\,20=40$ feet per second;

$$s=\frac{1}{2}ft^2=\frac{1}{2}\,.\,2\,.\,(20)^2=400 \text{ feet.}$$

3. 30 miles per hour $=44$ feet per second;

$$v=u+ft, \text{ i.e. } 44=4+t,$$

whence $$t=40 \text{ seconds.}$$

4. $s=\frac{1}{2}ft^2$, *i.e.* $1000=\frac{1}{2}f(10)^2$,

whence $$f=20 \text{ ft.-sec. units.}$$

5. $v=ft$, *i.e.* $30=3t$,

whence $$t=10 \text{ seconds;}$$

$$s=\frac{1}{2}ft^2, \text{ i.e. } s=\frac{1}{2}\,.\,3\,.\,(10)^2=150 \text{ centimetres.}$$

6. $v=u+ft$, *i.e.* $0=100-2t$,

so that $$t=50 \text{ seconds;}$$

$$s=ut+\frac{1}{2}ft^2=100\,.\,50-\frac{1}{2}\,.\,2\,.\,(50)^2=2500 \text{ centimetres}$$

$$=25 \text{ metres.}$$

7. Here $171 = \frac{1}{2}f \cdot (10)^2 - \frac{1}{2}f \cdot 9^2 = \frac{1}{2}f \cdot 19,$

so that $f = 18$ ft.-sec. units.

8. As in Ex. 2, p. 39, we have

$$8\tfrac{1}{2} = u + \frac{15}{2}f, \text{ and } 7\tfrac{1}{2} = u + \frac{25}{2}f$$

whence $u = 10$ feet per second,

and $f = -\frac{1}{5}$ ft.-sec. units;

the negative value of f shews that it is a retardation.

9. If u be the initial velocity, and f be the acceleration, we have

$$20\tfrac{1}{2} = u \cdot 1 + \frac{1}{2}f \cdot 1^2 \text{.........................(1)}$$

and $$20\tfrac{1}{2} + 23\tfrac{1}{2} = u \cdot 2 + \frac{1}{2}f \cdot 2^2 \text{.......................(2).}$$

By subtraction,

$$23\tfrac{1}{2} = u + \frac{1}{2}f \cdot 3 \text{.........................(3).}$$

Again, from (3) subtract (1), and we have

$$3 = \frac{1}{2}f \cdot 2, \text{ i.e. } f = 3 \text{ ft.-sec. units,}$$

and therefore

$$u = 23\tfrac{1}{2} - \frac{9}{2} = \frac{47-9}{2} = 19 \text{ feet per second.}$$

Again, $v^2 = 2fs$, where $v = 19$.

$$\therefore s = \frac{19 \times 19}{2 \times 3} = \frac{361}{6} = 60\tfrac{1}{6} \text{ feet.}$$

10. Let the time of motion be t seconds, so that

$$\frac{1}{2}ft^2 - \frac{1}{2}f(t-1)^2 = \frac{9}{25} \cdot \frac{1}{2}ft^2.$$

Hence $$\frac{16}{25}t^2 = (t-1)^2,$$

so that $$\frac{4}{5}t = t - 1, \text{ and } t = 5 \text{ seconds.}$$

Also $\frac{1}{2} = \frac{1}{2}f \cdot 1^2$, so that $f = 1$ ft.-sec. units;

hence the required distance

$$= \frac{1}{2}f \cdot 5^2 = 12\tfrac{1}{2} \text{ feet.}$$

11. We have

$$25=\left[u\cdot\frac{3}{2}+\frac{1}{2}f\cdot\left(\frac{3}{2}\right)^2\right]-\left(u\cdot 1+\frac{1}{2}f\cdot 1^2\right)=\frac{u}{2}+\frac{5f}{8}\ \ldots\ (1)$$

and

$$198=\left[u\cdot 11+\frac{1}{2}f\cdot(11)^2\right]-\left[u\cdot 10+\frac{1}{2}f\cdot(10)^2\right]=u+\frac{21f}{2}\ (2).$$

Solving (1) and (2), we have

$$u=30 \text{ feet per second, and } f=16 \text{ ft.-sec. units.}$$

12. During the last 3 seconds the velocity v

$$=\frac{72}{3}=24 \text{ feet per second,}$$

and therefore $u+3f=24$; also $81=3u+\frac{1}{2}f\cdot 3^2$, *i.e.* $u+\frac{3}{2}f=27$; solving, we have

$$u=30 \text{ feet per second,}$$

and $$f=-2 \text{ ft.-sec. units (a retardation).}$$

13. 40 miles per hour $=\dfrac{40\times 1760\times 3}{60\times 60}$ feet per second

$$=\frac{176}{3} \text{ feet per second;}$$

$$10 \text{ miles per hour}=\frac{44}{3} \text{ feet per second;}$$

$$150 \text{ yards}=450 \text{ feet.}$$

The formula

$$v^2=u^2-2fs \quad\text{gives}\quad f=\left[\left(\frac{176}{3}\right)^2-\left(\frac{44}{3}\right)^2\right]\div 2\cdot 450,$$

whence $$f=\frac{484}{135} \text{ ft.-sec. units;}$$

and then, by the relation $v^2=2fs$,

we have $$\left(\frac{44}{3}\right)^2=2\cdot\frac{484}{135}\cdot s, \quad\text{whence}\quad s=30 \text{ feet.}$$

14. If t_1, t_2 and t_3 be the required times respectively, we have

$$1=\frac{1}{2}f\cdot t_1^2=9t_1^2, \quad\text{so that}\quad t_1=\frac{1}{3}\text{ sec.;}$$

$$2=9(t_1+t_2)^2, \quad\text{so that}\quad t_1+t_2=\frac{\sqrt{2}}{3},$$

and $$t_2=\frac{\sqrt{2}-1}{3}\text{ sec.;}\quad 3=9(t_1+t_2+t_3)^2,$$

$$i.e.\quad t_1+t_2+t_3=\frac{\sqrt{3}}{3},\ \text{and therefore } t_3=\frac{\sqrt{3}-\sqrt{2}}{3}\ \text{sec.}$$

15. Let t be the number of seconds the second particle takes to overtake the first at a distance s from O. Then

$$s=4\,(t+2)\,;\ \text{also}\ s=5t+\frac{1}{2}\,.\,3\,.\,t^2.$$

Hence $$8t+16=10t+3t^2,\ i.e.\ 3t^2+2t-16=0.$$

$$\therefore\ (3t+8)\,(t-2)=0,\ i.e.\ t=2\ \text{seconds.}$$

Also $$s=4\,(t+2)=4\,.\,4=16\ \text{feet.}$$

16. Assume that the point is moving with a constant acceleration f, and that at the commencement of the first second its velocity is u.

We then have $$7=u+\frac{1}{2}f,$$

$$11=\left(3u+\frac{1}{2}f\,.\,3^2\right)-\left(2u+\frac{1}{2}f\,.\,2^2\right)=u+\frac{5}{2}f,$$

and $$17=\left(6u+\frac{1}{2}f\,.\,6^2\right)-\left(5u+\frac{1}{2}f\,.\,5^2\right)=u+\frac{11}{2}f.$$

These equations are satisfied by $u=6$ and $f=2$. Hence the equations are consistent.

17. The initial velocity is $3\sqrt{2}$ both in a north and an east direction. Hence, if x and y be the east and north displacements, we have

$$x=3\sqrt{2}\,.\,1+\frac{1}{2}\,.\,6=3\,(1+\sqrt{2}),$$

and $$y=3\sqrt{2}\,.\,1+\frac{1}{2}\,.\,8=4+3\sqrt{2}.$$

Hence $$\sqrt{x^2+y^2}=\sqrt{61+12\sqrt{2}},$$

and $$\tan\theta=\frac{y}{x}=\frac{2+\sqrt{2}}{3}.$$

18. Let t be the time so that $1500=200t-\frac{1}{2}\,.\,10t^2$.

$$\therefore\ t^2-40t+300=0,\ \text{so that}\ t=10\ \text{or}\ 30.$$

The value 10 secs. corresponds to the time when the particle first arrives at the given distance. It then goes on past this point and at the end of 20 secs. its velocity is just zero; it then returns and at the end of 10 secs. more is again at the given distance from the starting point.

19. Let x be the distance between the points at time t. Then

$$x=ut-\tfrac{1}{2}ft^2=\frac{u^2}{2f}-\tfrac{1}{2}f\left[t-\frac{u}{f}\right]^2.$$

Hence x is greatest when $\left[t-\frac{u}{f}\right]^2$ is least, *i.e.* when it vanishes, and then $t=\frac{u}{f}$ and $x=\frac{u^2}{2f}$.

20. It is assumed that during each minute the velocity alters uniformly from the value at the beginning to the value at the end of the minute.

The average velocities during these minutes are thus:

$$\frac{0+25}{2},\ \frac{25+40}{2},\ \frac{40+50}{2},\ \frac{50+50}{2}\ \ldots\ldots\ \frac{20+0}{2}$$

miles per hour.

The required average velocity

$$=\text{sum of these}\div 12$$

$$=\frac{1}{24}[2\,(25+40+50+50+45+40+40+45+45+35+20)]$$

$$=\frac{435}{12}=36\tfrac{1}{4}\text{ miles per hour.}$$

21. As in the last question the space described

$$=\frac{0+5}{2}+\frac{5+18}{2}+\frac{18+38}{2}+\frac{38+62}{2}+\frac{62+78}{2}+\frac{78+81}{2}+\frac{81+83}{2}$$

$$=2\tfrac{1}{2}+11\tfrac{1}{2}+28+50+70+79\tfrac{1}{2}+82=323\tfrac{1}{2}.$$

Also the increase of velocity is clearly greatest in the fourth second and then $=24$ ft. per sec.; $\therefore$ required greatest acceleration $=24$.

22. The average velocities during the successive periods of 5 secs. are 2, 6·4, 13·9, 20·5, 18·85, and 12·85, and the distances described are therefore 10, 32, 69·5, 102·5, 94·25, and 64·25 feet, the sum of which $=372·5$. Also from the 15th to the 20th sec. the increase in the velocity is 3 and therefore the acceleration $=\frac{3}{5}$ ft.-sec. units.

EXAMPLES. VI. (Pages 46—48.)

1. (1) The required height h is given by $(40)^2=2gh$.

$$\therefore\ h=\frac{40\times 40}{2\times 32}=25\text{ feet.}$$

(2) The required times t are given by

$$9=40t-\frac{1}{2}gt^2,\ \textit{i.e.}\ 16t^2-40t+9=0,$$

whence $$t=\frac{1}{4}\text{ sec. and }\frac{9}{4}\text{ secs.}$$

2. The required times t are given by

(i) $25=40-gt$, whence $t=\frac{15}{32}$ sec.

(ii) $25=40t-\frac{1}{2}gt^2$, *i.e.* $16t^2-40t+25=0$,

i.e. $(4t-5)^2=0$, whence $t=1\frac{1}{4}$ sec.;

the values of t in this case being equal, the particle is at its highest point.

3. The velocity of 20 feet per second may be upwards or downwards, hence $\pm 20=60-gt$,

$$\therefore\ t=\frac{1}{32}(60\mp 20)=1\tfrac{1}{4}\text{ sec., or }2\tfrac{1}{2}\text{ secs.}$$

The corresponding heights are

$$60\times 1\tfrac{1}{4}\ \ \frac{1}{2}g\left(\frac{5}{4}\right)^2,\text{ and }60\times 2\tfrac{1}{2}-\frac{1}{2}g\left(\frac{5}{2}\right)^2,$$

i.e. 50 feet, and 50 feet. This is as it should be, since the velocity is the same in magnitude at the same height, whether the motion is upwards or downwards.

4. (1) $s=\frac{1}{2}gt^2=\frac{1}{2}\cdot 32\cdot(10)^2=1600$ feet,

(2) $10=\frac{1}{2}gt^2$, whence $t=\frac{\sqrt{10}}{4}$ seconds.

(3) $\pm 1000=10u-\frac{1}{2}g\cdot(10)^2$,

whence $u=160\pm 100=260$ feet per second,

or 60 feet per second, both upwards.

5. If x feet be the required depth, we have

$$x=96\times 3+\frac{1}{2}g\cdot(3)^2=288+144=432\text{ feet.}$$

6. Measuring upwards from the bottom of the mine, since

$$s=ut-\frac{1}{2}gt^2,\ \text{ we have }\ 88g=24gt-\frac{1}{2}gt^2,$$

i.e. $t^2-48t+176=0$, *i.e.* $(t-44)(t-4)=0$.

Also the greater value of t is required, so that $t=44$ seconds.

7. If u be the velocity of projection, and t seconds be the required time, $u^2=2g\times 225=64\times 225$.

$$\therefore\ u=8\times 15=120\text{ feet per second.}$$

$$\therefore\ 176=ut-\frac{1}{2}gt^2=120t-16t^2,$$

i.e. $(4t-15)^2=225-176=49$, whence $t=2$ seconds or $5\frac{1}{2}$ seconds.

8. The initial velocity u is given by

$$(436)^2 = u^2 - 2 \,.\, 981 \,.\, \frac{545}{10} = u^2 - 981 \times 109.$$

$$\therefore \; u^2 = (109)^2 (4^2 + 9) = (109)^2 \,.\, 5^2,$$

so that $u = 545$ centimetres per second.

The required time

$$= \frac{436}{g} = \frac{4 \times 109}{981} = \frac{4}{9} \text{ second.}$$

9. Suppose t seconds before, so that the velocity of 5000 centimetres per second upwards in time t has become 5000 centimetres per second downwards; hence $\quad -5000 = 5000 - gt,$

i.e. $$t = \frac{10000}{981} = 10{\cdot}2 \text{ seconds.}$$

10. If it rise h centimetres,

$$(6540)^2 = 2gh.$$

$$\therefore \; h = \frac{(6540)^2}{2 \times 981} = \frac{(2180)^2}{2 \times 109} = 2180 \times \frac{20}{2} = 21800 \text{ centimetres} = 218 \text{ metres.}$$

Also the body moves upwards till its velocity is zero, and therefore for a time t seconds given by

$$t = \frac{6540}{g} = \frac{6540}{981} = \frac{60}{9} = \frac{20}{3} = 6\tfrac{2}{3} \text{ seconds.}$$

11. Here we have $176{\cdot}99 = \frac{1}{2} g (6^2 - 5^2) = \frac{11g}{2}$,

so that $$g = \frac{2 \times 176{\cdot}99}{11} = 2 \times 16{\cdot}09 = 32{\cdot}18.$$

12. Here we have

$$224 = \frac{1}{2} g \left[t^2 - (t-1)^2\right] = 16\,(2t - 1),$$

whence $t = 7\frac{1}{2}$ seconds.

Also the required height

$$= \frac{1}{2} gt^2 = 16 \left(\frac{15}{2}\right)^2 = 900 \text{ feet.}$$

13. If t seconds be the time of falling, we have

$$\frac{1}{2} gt^2 - \frac{1}{2} g\,(t-1)^2 = \frac{16}{25} \cdot \frac{1}{2} gt^2,$$

so that $$(t-1)^2 = \frac{9}{25} t^2, \; i.e. \; t - 1 = \frac{3}{5} t,$$

i.e. $t = 2\frac{1}{2}$ seconds.

Hence the required height

$$= \frac{1}{2} \cdot g \cdot \left(\frac{5}{2}\right)^2 = 100 \text{ feet.}$$

14. Proceed as in the last example.

15. If h be the required height, we have $(96)^2=2gh$, so that $h=144$ feet. If A overtake B in t seconds more, B has fallen through a distance $\frac{1}{2}gt^2$, and therefore also A. Hence we have

$$96(4+t)-\frac{1}{2}g(4+t)^2=-\frac{1}{2}gt^2,$$

whence $t=4$ seconds.

16. When at 960 feet from the ground, the body loses its velocity in 2 seconds, and its velocity then is $2g$, *i.e.* 64 feet per second. Hence we have

$$(64)^2=u^2-2g\,.\,960,$$

so that

$$u^2=(64)^2\times 16,$$

i.e.

$$u=256 \text{ feet per second.}$$

Also the whole height ascended

$$=\frac{u^2}{2g}=\frac{(256)^2}{64}=1024 \text{ feet.}$$

17. We have

$$720=ut+\frac{1}{2}gt^2,$$

and

$$2240=u\,.\,2t+\frac{1}{2}g\,.\,(2t)^2.$$

$$\therefore\ 2240-2gt^2=2ut=2\left(720-\frac{1}{2}gt^2\right)=1440-gt^2.$$

$$\therefore\ t=\sqrt{800\div g}=5 \text{ seconds.}$$

Also, substituting for t, we have $u=64$ feet per second.

18. If x feet be the depth of the well, and t seconds be the time of fall,

$$x=\frac{1}{2}gt^2,\ \textit{i.e.}\ t=\sqrt{\frac{2x}{g}},$$

and the time after that taken by the sound to reach the surface

$$=\frac{x}{1120} \text{ seconds.}$$

Hence we have

$$\sqrt{\frac{2x}{g}}+\frac{x}{1120}=7\tfrac{7}{10},\ \textit{i.e.}\ \frac{x}{1120}+\frac{\sqrt{x}}{4}=\frac{77}{10}.$$

$$\therefore\ x+280\sqrt{x}=77\times 112,$$

i.e.

$$(\sqrt{x}-28)(\sqrt{x}+308)=0.$$

$$\therefore\ \sqrt{x}=28, \text{ and } x=784 \text{ feet;}$$

the second solution is inadmissible, since $\sqrt{\frac{2x}{g}}$ must be positive.

19. The depth of the well

$$=(96)^2\div 2g=144 \text{ feet}.$$

Hence, if v be the velocity of sound, we have

$$3\tfrac{9}{70}=\frac{144}{v}+\sqrt{\frac{2\,.\,144}{g}}=\frac{144}{v}+3;$$

$$\therefore\ v=\frac{70}{9}\times 144=1120 \text{ feet per second}.$$

20. If h be the required height, we have

$$(40)^2=2\,.\,\frac{g}{6}\,.\,h=\frac{32}{3}h, \text{ whence } h=150 \text{ feet}.$$

EXAMPLES. VII. (Page 50.)

1. If l be the length of the plane described, and t seconds be the time of describing it, we have

$$(80)^2=2g\,.\sin 30^\circ\,.\,l=32l,$$

whence $l=200$ feet. Also,

$$80=g\sin 30^\circ\,.\,t=16t,$$

whence $t=5$ seconds.

2. If v be the required velocity and t seconds be the required time, we have

$$v=\sqrt{2g\,.\,12}=16\sqrt{3} \text{ feet per second};$$

acceleration down the plane

$$=g\sin\alpha=32\times\frac{12}{15}=\frac{128}{5};$$

hence we have $16\sqrt{3}=\frac{128}{5}t$, *i.e.* $t=\frac{5\sqrt{3}}{8}$ seconds.

$$\left[\textit{or}\ 15=\frac{1}{2}\,.\,\frac{128}{5}t^2, \text{ whence } t=\frac{5\sqrt{3}}{8} \text{ seconds.}\right]$$

3. If α be the required inclination, we have

$$(16\sqrt{2})^2=2g\sin\alpha\times 16, \text{ so that } 16=32\sin\alpha,$$

$$\textit{i.e.}\ \sin\alpha=\frac{1}{2},\ \textit{i.e.}\ \alpha=30^\circ.$$

4. If α be the inclination of the plane,

$$\sin\alpha=\frac{\text{height}}{\text{length}}=\frac{h}{l} \text{ say} \ldots\ldots\ldots\ldots (1);$$

and the time t seconds of sliding down is given by

$$l=\frac{1}{2}g\sin\alpha\,.\,t^2 \qquad \text{.................. .. ...(2)};$$

also the time in falling freely is given by

$$h=\frac{1}{2}g\,.\left(\frac{t}{4}\right)^2 \qquad \text{.. . ...(3)}.$$

From (2) and (3) by division, we have

$$\frac{h}{l}=\frac{1}{16\sin\alpha};\ \text{hence by (1)},\ \frac{h}{l}=\frac{l}{16h},$$

and therefore $\qquad \dfrac{h^2}{l^2}=\dfrac{1}{16}$, *i.e.* $h:l=1:4$.

5. If s be the required spaces, v be the required velocities, and

$$\sin\alpha=\frac{3}{5},$$

we have

(1) $\quad s=16\times4-\dfrac{1}{2}g\sin\alpha\,.\,4^2=64-16\,.\,\dfrac{3}{5}\,.\,16=-89\tfrac{3}{5}$ feet,

and $\quad v=16-g\sin\alpha\,.\,4=16-128\,.\,\dfrac{3}{5}=-60\tfrac{4}{5}$ feet per second.

(2) $\quad s=16\times4+\dfrac{1}{2}g\sin\alpha\,.\,4^2=217\tfrac{3}{5}$ feet,

and $\quad v=16+g\sin\alpha\,.\,4=92\tfrac{4}{5}$ feet per second.

6. If α be the inclination of the plane, we have

$$2207{\cdot}25=\frac{1}{2}g\sin\alpha\,(5^2-4^2),$$

i.e. $\qquad \dfrac{8829}{4}=\dfrac{981\times9}{2}\sin\alpha,$

whence $\qquad \sin\alpha=\dfrac{1}{2}$, *i.e.* $\alpha=30°$.

7. If a be the diameter of the circle, the time down AB

$$=\sqrt{\frac{2a}{g}}.$$

Hence we have

$$a=\frac{1}{2}g\cos\theta\left(2\sqrt{\frac{2a}{g}}\right)^2=4a\cos\theta,$$

so that $\qquad \cos\theta=\dfrac{1}{4}$, *i.e.* $\theta=\cos^{-1}\dfrac{1}{4}=75°\ 31'$.

EXAMPLES. VIII. (Pages 54—57.)

1. Suppose at height h feet; then we have

$$-h = ut - \frac{1}{2}gt^2 = 32 \times 17 - 16\,(17)^2,$$

i.e.
$$h = 16 \times 17\,(17 - 2) = 4080 \text{ feet}.$$

2. If t seconds be the required time, then the distance descended by the first body when they meet

$$= \frac{1}{2}gt^2 = 16t^2;$$

and the distance ascended by the second body

$$= ut - \frac{1}{2}gt^2 = 64t - 16t^2;$$

also the sum of these distances $= 64$ feet, *i.e.*

$$64 = 16t^2 + 64t - 16t^2 = 64t,$$

so that $t = 1$ second.

In the second case, the first body will only descend a distance $16\,(t-1)^2$, so that we have

$$64 = 16\,(t-1)^2 + 64t - 16t^2 = 32t + 16;$$

hence
$$32t = 64 - 16 = 48,$$

and
$$t = 1\tfrac{1}{2} \text{ second}.$$

3. Let t seconds be the time before they meet, and u be the required initial velocity. Then

$$144 = ut - \frac{1}{2}gt^2, \text{ and } 144 = \frac{1}{2}gt^2.$$

Hence $t = 3$ seconds, and $u = 96$ feet per second.

Also the required velocity of the projected body

$$= u - gt = 96 - 32 \times 3 = 0.$$

4. Let h be the height of the tower, so that the initial velocity of the projected body is $\sqrt{2gh}$. Let t be the time that elapses before the bodies meet. Then $h =$ the sum of the distances described by the two bodies in time t

$$= \left(\sqrt{2gh}\,.\,t - \frac{1}{2}gt^2\right) + \frac{1}{2}gt^2 = \sqrt{2gh}\,.\,t;$$

hence
$$t = \sqrt{\frac{h}{2g}},$$

and the first body will have fallen through a distance

$$\frac{1}{2}gt^2, \text{ i.e. } \frac{1}{2}g\,.\,\frac{h}{2g}, \text{ i.e. } \frac{h}{4}.$$

5. Let u be the velocity of projection of the second particle, and let the particles meet at the end of time t. Then we have

$$\frac{2}{3}h=\frac{1}{2}gt^2,$$

and

$$\frac{1}{3}h=ut-\frac{1}{2}gt^2.$$

Hence, by addition, $h=ut$ and therefore

$$h=\frac{3}{4}gt^2=\frac{3}{4}\frac{gh^2}{u^2}\text{, so that } u^2=\frac{3}{4}gh.$$

Hence the height required

$$=\frac{u^2}{2g}=\frac{3}{8}h.$$

6. Let l be the length of the plane, a be its inclination to the horizon, u be the required velocity of projection, and t be the time that elapses before the bodies meet. Then

$$\frac{1}{2}l=\frac{1}{2}g\sin a\,.\,t^2 \text{ ....(1),}$$

and

$$\frac{1}{2}l=ut-\frac{1}{2}g\sin a\,.\,t^2 \text{ ....(2).}$$

Hence, adding, we have

$$l=ut,\ i.e.\ t=\frac{u}{l}.$$

Substituting in (1) we have

$$u^2=gl\sin a=gh,$$

where h is the height of the plane. Also the velocity of the first body at the instant when the bodies meet

$$=\sqrt{2g\,.\,\frac{h}{2}}=\sqrt{gh},$$

and the velocity of the second body

$$=\sqrt{u^2-2g\,.\,\frac{h}{2}}=\sqrt{gh-gh}=0.$$

7. Let the bodies meet T seconds after the first particle starts, and at a height h; then

$$h=uT-\frac{1}{2}gT^2,\text{ and } h=u(T-t)-\frac{1}{2}g(T-t)^2.$$

$$\therefore\ uT-\frac{1}{2}gT^2=uT-ut-\frac{1}{2}gT^2+gtT-\frac{1}{2}gt^2,$$

whence

$$T=\frac{u}{g}+\frac{1}{2}t=\frac{1}{g}\left(u+\frac{1}{2}gt\right).$$

Also
$$h=uT-\frac{1}{2}gT^2=T\left(u-\frac{1}{2}gT\right)$$
$$=\frac{1}{g}\left(u+\frac{1}{2}gt\right)\left[u-\frac{1}{2}\left(u+\frac{1}{2}gt\right)\right]=\frac{1}{2g}\left(u^2-\frac{1}{4}g^2t^2\right).$$

8. The height of the balloon at the end of 30 seconds
$$=\frac{1}{2}\cdot 4\cdot(30)^2=1800 \text{ feet},$$
and its velocity then is 4×30, *i.e.* 120 feet per second.

Hence, if t seconds be the required time, we have
$$-1800=120t-\frac{1}{2}gt^2,$$
whence $t=15$ seconds.

9. The velocity on reaching the pane of glass
$$=gt=32\times 5=160 \text{ feet per second};$$
the velocity after passing through the glass $=80$ feet per second; hence the height of the glass above the ground
$$=ut+\frac{1}{2}gt^2=80\cdot 1+\frac{1}{2}\cdot 32\cdot 1=96 \text{ feet}.$$

10. If the body fall for t seconds, we have
$$\frac{1}{2}g\,[t^2-(t-1)^2]:\frac{1}{2}g\,[(t-1)^2-(t-2)^2]=3:2.$$
$$\therefore\ 2\,(2t-1)=3\,(2t-3),\ \text{whence}\ t=3\tfrac{1}{2}\ \text{seconds}$$
Hence the required height
$$=\frac{1}{2}gt^2=\frac{1}{2}\cdot 32\cdot\left(\frac{7}{2}\right)^2=196 \text{ feet},$$
and the required velocity
$$=gt=32\cdot\frac{7}{2}=112 \text{ feet per second}.$$

11. If α be the inclination of the plane,
$$\sin\alpha=\frac{64}{288},$$
and the whole time t of descending the plane is given by
$$288=\frac{1}{2}g\sin\alpha\cdot t^2,$$
whence $t=9$ seconds, so that the equal times are each 3 seconds. Hence the first part is described in 3 seconds, and
$$=\frac{1}{2}g\sin\alpha\cdot t^2=16\cdot\frac{64}{288}\cdot 9=32 \text{ feet};$$

the first two parts are described in 6 seconds, and

$$=16 \cdot \frac{64}{288} \cdot 36 = 128 \text{ feet},$$

so that the second part

$$=(128-32) \text{ feet} = 96 \text{ feet},$$

and therefore the third part

$$=288-(32+96)=160 \text{ feet}.$$

12. If AB be the chord, AC be a horizontal radius, and θ be the angle AB makes with the vertical, then the $\angle BAC = 90° - \theta$, and the $\angle ACB = 2\theta$. Also if t be the time down AB, we have

$$AB = \frac{1}{2} g \cos\theta \cdot t^2,$$

also $$AB = 2AC \sin\theta;$$

hence $$\frac{1}{2} gt^2 = 2AC \tan\theta, \text{ i.e. } t \text{ varies as } \sqrt{\tan\theta}.$$

13. If AB be any one of the rods at an angle θ to the vertical AO, and the ring on AB in time t reach B, and if BO be perpendicular to AB, then

$$AB = \frac{1}{2} g \cos\theta \cdot t^2, \text{ and } AO = AB \sec\theta = \frac{1}{2} gt^2,$$

so that AO is constant in magnitude and position; also OBA is a right angle so that B lies on the sphere described on AO as diameter; thus the ring at B after a time t lies on a sphere of radius $\frac{gt^2}{4}$; similarly for any other of the rings.

14. Since the heights of the planes are the same, it follows, by Art 47, that the velocities acquired are the same.

15. Since the bodies have descended the same vertical distance, the velocities acquired are the same. Also, if h, l and θ be respectively the height, length and inclination of the plane, and t_1 and t_2 be the times taken, we have

$$\frac{h}{l} = \frac{\frac{1}{2}gt_1^2}{\frac{1}{2}g \sin\theta \cdot t_2^2} = \frac{1}{\sin\theta} \cdot \frac{t_1^2}{t_2^2} = \frac{l}{h} \cdot \frac{t_1^2}{t_2^2},$$

so that $$\frac{t_1}{t_2} = \frac{h}{l}.$$

16. If h be the given height and θ be the inclination of the plane to the vertical, the length of the plane is $\frac{h}{\cos\theta}$, and the acceleration is $g\cos\theta$.

Hence $$\frac{h}{\cos\theta} = \frac{1}{2} g \cos\theta \cdot t^2,$$

so that $$t^2 \propto \sec^2\theta, \text{ i.e. } t \propto \sec\theta.$$

17. If $2a$ be the diameter of the circle, the length of the chord inclined at an angle θ to the vertical is $2a\cos\theta$. Hence, if v be the required velocity, we have

$$v^2=2g\cos\theta\,.\,2a\cos\theta=\frac{g}{a}(2a\cos\theta)^2,$$

so that $v \propto 2a\cos\theta \propto$ chord.

18. If $2a$ and $2b$ be the diameters of the two circles, and θ be the inclination of the chord to the vertical, the length of the intercepted portion is $(2a-2b)\cos\theta$.

Hence $$(2a-2b)\cos\theta=\frac{1}{2}g\cos\theta\,.\,t^2,$$

so that $t=2\sqrt{\dfrac{a-b}{g}}$, and is therefore constant.

19. If AB be the line of greatest slope, and AC be the groove, the component of gravity parallel to the plane is $g\sin\alpha$ along AB; therefore the component along AC

$$=g\sin\alpha\cos\beta;$$

also, if t be the required time, we have

$$AC=\frac{1}{2}g\sin\alpha\cos\beta\,.\,t^2;$$

but $$AC=AB\sec\beta=h\operatorname{cosec}\alpha\sec\beta;$$

hence $$h\operatorname{cosec}\alpha\sec\beta=\frac{1}{2}g\sin\alpha\cos\beta\,.\,t^2.$$

$$\therefore\ t=\sqrt{\frac{2h}{g}}\,.\operatorname{cosec}\alpha\sec\beta.$$

20. Let $v_1, v_2, \ldots\ldots v_n$ be the velocities of the point when it has described distances from rest equal to

$$\frac{s}{n},\ \frac{2s}{n}\ldots\ldots s$$

respectively. Then we have

$$v_1^2=2f\,.\,\frac{s}{n};\ v_2^2=v_1^2+2\left(f+\frac{f}{n}\right)\frac{s}{n};\ v_3^2=v_2^2+2\left(f+\frac{2f}{n}\right)\frac{s}{n};$$

$$\ldots\ldots v_n^2=v_{n-1}^2+2\left[f+\frac{(n-1)f}{n}\right]\frac{s}{n}.$$

Hence, by addition,

$$v_n^2=2f\frac{s}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)+\ldots\ldots\text{ to } n \text{ terms}\right]$$

$$=2f\frac{s}{n}\left[\frac{n}{2}\left\{2+(n-1)\frac{1}{n}\right\}\right]=fs\left(3-\frac{1}{n}\right).$$

$$\therefore\ v_n=\sqrt{fs\left(3-\frac{1}{n}\right)}.$$

21. Let $v_1, v_2, \ldots\ldots v_n$ be the velocities at the ends of successive intervals t, so that

$$v_1=ft, \quad v_2=v_1+2ft, \quad v_3=v_2+3ft, \ldots\ldots v_n=v_{n-1}+nft.$$

Hence, by addition,

$$v_n=ft(1+2+\ldots\ldots+n)=\frac{n(n+1)}{2}ft.$$

Also let $s_1, s_2 \ldots\ldots s_n$ be the distances described in these successive intervals, so that

$$s_1=\frac{1}{2}ft^2, \quad s_2=v_1t+\frac{1}{2}\cdot 2ft^2, \ldots\ldots s_n=v_{n-1}t+\frac{1}{2}\cdot nf\cdot t^2,$$

i.e. $$s_1=\frac{1}{2}ft^2, \quad s_2=\frac{1}{2}\cdot 4ft^2, \ldots\ldots s_n=\frac{1}{2}\cdot n^2ft^2.$$

Hence the total distance described

$$=\frac{1}{2}ft^2(1^2+2^2+\ldots\ldots+n^2)=\frac{n(n+1)(2n+1)}{12}\cdot ft^2.$$

22. If f be the acceleration, and a, b and c be the respective distances, we have

$$b=\frac{1}{2}fn^2, \text{ and } c=\frac{1}{2}f(n+1)^2.$$

$$\therefore b+c=\frac{1}{2}f(2n^2+2n+1).$$

Also $$a=\frac{1}{2}f[(n^2+n+1)^2-(n^2+n)^2]=\frac{1}{2}f(2n^2+2n+1).$$

Hence $$a=b+c.$$

23. Let h be the distance fallen through, and g_1 and g_2 be the numerical values of the acceleration due to gravity at the two places. The times of falling are

$$\sqrt{\frac{2h}{g_1}} \text{ and } \sqrt{\frac{2h}{g_2}};$$

also the velocities acquired are

$$\sqrt{2g_1h} \text{ and } \sqrt{2g_2h}.$$

Hence $$n=\sqrt{\frac{2h}{g_2}}-\sqrt{\frac{2h}{g_1}}=\frac{\sqrt{2g_1h}-\sqrt{2g_2h}}{\sqrt{g_1g_2}}=\frac{m}{\sqrt{g_1g_2}}.$$

$$\therefore \sqrt{g_1g_2}=\frac{m}{n}.$$

24. Let f_1 and $-f_2$ be the acceleration and retardation respectively, and let v be the maximum velocity.

Since the velocity which is acquired in the first part of the journey is lost in the second part, we have

$$3520f_1=\frac{v^2}{2}=1760f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also the sum of the times taken to create and destroy the velocity v is 3 minutes. *

See reverse Page (253)

$$\therefore \frac{v}{f_1}+\frac{v}{f_2}=180 \quad\text{.........................(2).}$$

From (1), $f_2=2f_1$. Hence, from (2), $v=120f_1$.
Substituting in (1), we have

$$f_1=\frac{22}{45},\quad f_2=\frac{44}{45},$$

and $$v=\frac{176}{3}\text{ feet per second}=40\text{ miles per hour.}$$

25. Let u be the initial velocity, and $-f$ the acceleration, so that

$$87=u-\frac{1}{2}f,\quad\text{and}\quad 85=u-\frac{3}{2}f.$$

Hence $u=88$ feet per second, and $f=2$.

The train will therefore come to rest in $\frac{u}{f}$, *i.e.* in 44, seconds; the distance it will have described is $\frac{u^2}{2f}$, *i.e.* 1936, feet. The time the train will take to pass the spectator is the difference between the times it takes to describe 1740 feet and 1452 feet. These times t_1 and t_2 are given by

$$1740=88t_1-t_1^2,\quad\text{and}\quad 1452=88t_2-t_2^2.$$

Hence, taking the smaller values, we have $t_1=30$, and $t_2=22$, so that the required time is 8 seconds.

26. Let x be the distance described in time t before the brakes are applied and let v be the velocity then.

$$\therefore\ 2fx=v^2=2f'(a-x) \quad\text{.....................(1),}$$

and $$ft=v=f'(T-t) \quad\text{..................(2),}$$

where T is the total time required.

From (1), $$v^2\left(\frac{1}{f}+\frac{1}{f'}\right)=2a,$$

and, from (2), $$v\left(\frac{1}{f}+\frac{1}{f'}\right)=T.$$

$$\therefore\ \frac{T^2}{2a}=\frac{1}{f}+\frac{1}{f'};\ \therefore\ \text{etc.}$$

27. Let $4a$ be the distance between the stations, f and f' the acceleration and retardation, and v the greatest speed. Then

$$2fa=v^2=2f'a \quad\text{.............................(1)}$$

and the time $$=\frac{v}{f}+\frac{2a}{v}+\frac{v}{f'}=\frac{6a}{v},\ \text{from (1).}$$

$$\therefore\ \text{average speed}=\frac{\text{total space}}{\text{time}}=4a\div\frac{6a}{v}=\frac{2v}{3}.$$

28. Let f = acceleration in the first part, x the distance described in time t before the upward force ceased, and v the maximum velocity.

Then $$2fx = v^2 = 2g(600 - x),$$
and $$ft = v = g(30 - t).$$

$$\therefore\ v^2\left(\frac{1}{f}+\frac{1}{g}\right) = 1200,\ \text{and}\ v\left(\frac{1}{f}+\frac{1}{g}\right) = 30.$$

$$\therefore\ \frac{30^2}{1200} = \frac{1}{f}+\frac{1}{g},\ \textit{i.e.}\ \frac{1}{f} = \frac{3}{4} - \frac{1}{32} = \frac{23}{32},\ \text{and}\ \therefore\ f = \frac{32}{23}.$$

Also $$v = 30 \div \left(\frac{1}{f}+\frac{1}{g}\right) = 30 + \frac{24}{32} = 40.$$

29. 50 miles per hour $= \frac{220}{3}$ ft. per sec.;

acceleration $$= \frac{220}{3} \div 300 = \frac{11}{45};$$

retardation $$= \left(\frac{220}{3}\right)^2 \div (2 \times 2640) = \frac{55}{54}.$$

Average speed during first and last portions = 25 miles per hour.

Also length of first portion
$$= \frac{1}{2}\cdot\frac{11}{45}\cdot 300^2 \text{ ft.} = \frac{25}{12} \text{ miles.}$$

Therefore total time
$$= \left(\frac{\frac{25}{12}}{25} + \frac{100 - \frac{25}{12} - \frac{1}{2}}{50} + \frac{\frac{1}{2}}{25}\right) \text{hours} = 2 \text{ hrs. } 8\tfrac{1}{10} \text{ mins.}$$

EXAMPLES. IX. (Pages 66 – 68.)

4. (1) In the equation $P = mf$, put $P = 5$ and $m = 10$; then
$$f = \frac{5}{10} = \frac{1}{2} \text{ ft.-sec. units.}$$

(2) The weight of 5 lbs. $= 5g$ poundals; hence $5g = 10f$, and
$$f = \frac{g}{2} \text{ ft.-sec. units.}$$

(3) Here $$50g = 10 \times 2240f,$$
so that $$f = \frac{g}{448} \text{ ft.-sec. units.}$$

5. (1) $P = (20 \times 10)$ poundals $= 200$ poundals.

(2) $\frac{200}{g}$ lbs. wt. $= \frac{200}{32} = 6\frac{1}{4}$ lbs. wt.

6. $$f=\frac{15}{5}=3,$$

so that $$P=(160\times 3)\text{ poundals}=\frac{160\times 3}{32}\text{ lbs. wt.}=15\text{ lbs. wt.}$$

7. 3 miles per hour $=\dfrac{3\times 1760\times 3}{60\times 60}=\dfrac{22}{5}$ feet per second; hence

$$f=\frac{22}{5}\div 10=\frac{11}{25}\text{ ft.-sec. units,}$$

and $$P=\left(1120\times\frac{11}{25}\right)\text{ poundals}=\frac{224\times 11}{5\times 32}\text{ lbs. wt.}=15\tfrac{2}{5}\text{ lbs. wt.}$$

8. The moving force = the weight of 2 lbs. $=2g$ poundals, and the mass moved = 40 lbs.; hence

$$f=\frac{2g}{40}=\frac{8}{5}\text{ ft.-sec. units;}$$

therefore the required velocity

$$=ft=\frac{8}{5}\times 30=48\text{ feet per second,}$$

and the required distance

$$=\frac{1}{2}ft^2=\frac{1}{2}\cdot\frac{8}{5}\cdot(30)^2=720\text{ feet.}$$

9. We have $$700=\frac{1}{2}f\cdot 10^2,$$

whence $$f=14\text{ cm.-sec. units;}$$

also $$\text{force : weight}=f:g=14:981;$$

and the required velocity

$$=ft=14\times 10=140\text{ cms. per second.}$$

10. If f be the acceleration, then $g=18f$,

$$i.e.\ f=\frac{32}{18}=\frac{16}{9};$$

hence, if t be the required time and v be the required velocity, we have

$$50=\frac{1}{2}ft^2=\frac{8}{9}t^2,$$

whence $t=7\frac{1}{2}$ seconds, and

$$v=ft=\frac{16}{9}\times\frac{15}{2}=13\tfrac{1}{3}\text{ feet per second.}$$

11. We have $112000 = 200 \times 2240 \times f$, so that

$$f = \frac{1}{4} \text{ ft.-sec. units.}$$

Now 30 miles per hour = 44 feet per second, so that

$$44 = ft = \frac{t}{4},$$

i.e. $t = (44 \times 4)$ seconds = 176 seconds = 2 minutes 56 seconds.

12. We have $$f = \frac{10g}{2240} = \frac{1}{7}.$$

Hence $$14 = \frac{1}{2} ft^2 = \frac{t^2}{14}, \text{ so that } t = 14 \text{ seconds.}$$

13. The acceleration f is given by

$$50 = \frac{1}{2} f \cdot 5^2, \text{ i.e. } f = 4.$$

Hence the required pressure P

$$= mf = 224 \times 4 \text{ poundals} = \frac{224 \times 4}{32} \text{ lbs. wt} = 28 \text{ lbs. wt.}$$

14. Here we have

$$112g = 16 \times 2240f, \text{ whence } f = \frac{1}{10}.$$

Hence the required distance $= \frac{1}{2} f \cdot (60)^2$ feet = 180 feet.

15. $P = 10g = 10 \times 981 = 27f$, *i.e.* $f = \frac{1090}{3}$ C.G.S. units.

Hence the required velocity

$$= \frac{1090}{3} \times 1 = 363\tfrac{1}{3} \text{ centimetres per second;}$$

and the distance travelled in the first second

$$= \frac{1}{2} f \cdot 1^2 = 181\tfrac{2}{3} \text{ centimetres.}$$

Also with velocity $\frac{1090}{3}$ C.G.S. units the body will travel in the next minute a distance

$$= 60 \times \frac{1090}{3} = 21800 \text{ centimetres} = 218 \text{ metres.}$$

16. The acceleration is given by

$$1000 = \frac{1}{2} f \cdot (10)^2,$$

so that $f = 20$. Hence $1000g = m \cdot 20$, so that

$$m = (50 \times 981) \text{ grammes} = \frac{98 \cdot 1}{2} \text{ kilogrammes} = 49 \cdot 05 \text{ kilogrammes.}$$

17. Here $(10)^2 = 2 . f . 25$, so that $f = 2$.

Hence $9g = m . 2$, *i.e.* $m = 144$ lbs.

18. Here $48 = 3f$, *i.e.* $f = 16$.

Hence $6g = m . 16$, *i.e.* $m = 12$ lbs.

19. (1) We have $100 = 2f$, so that the retardation $f = 50$. Hence the required force $P = (3 \times 50)$ poundals + the weight of the body

$$= \left(\frac{150}{32} + 3\right) \text{ lbs. wt.} = 7\tfrac{11}{16} \text{ lbs. wt.}$$

(2) Since $v^2 = 2fs$, we have

$$f = \frac{(100)^2}{2 \times 2} = 2500.$$

Hence the required force

$P = (3 \times 2500)$ poundals + the weight of the body

$$= \left(\frac{7500}{32} + 3\right) \text{ lbs. wt.} = 237\tfrac{3}{8} \text{ lbs. wt.}$$

20. If P and P' be the forces in poundals, and f and f' be the accelerations they produce, we have

$$5 = \frac{1}{11} f, \text{ i.e. } f = 55;$$

and, since 18 miles per hour $= \dfrac{132}{5}$ feet per second,

$$\frac{132}{5} = 60f', \text{ i.e. } f' = \frac{11}{25}.$$

Therefore $P = 5f = 5 \times 55$,

and $P' = 625f' = 625 \times \dfrac{11}{25} = 5 \times 55$; hence $P = P'$.

21. Let P be the pressure of the sand in poundals, so that the total force on the mass is $P - 10g$. Also the acceleration produced being f, we have $P - 10g = 10f$; and also since the velocity of the mass on first touching the sand is $\sqrt{640}$, $640 = 2f . 1$. Hence $f = 320$, and therefore

$$P = 110g = 110 \text{ lbs. wt.}$$

22. We have $(45000)^2 = 2f . 200$, so that

$$f = \frac{(45)^2 \times (10)^4}{4}.$$

Hence the mean force

$$= 1000f = (10)^7 \times \frac{2025}{4} = 5{\cdot}0625 \times (10)^9 \text{ dynes.}$$

23. If s feet be the length of the bore, f be the acceleration, and P be the total pressure in poundals, we have

$$(1490)^2=2fs, \text{ and } (1330)^2=2f(s-1);$$

hence, by subtraction, we have $f=1410\times 160$, and

$$P=100f=(1410\times 160\times 100) \text{ poundals}=315 \text{ tons wt. nearly.}$$

24. The retardation f produced by the resistance is given by

$$(200)^2=2\,.\,f\,.\,\frac{3}{4}.$$

Hence, if v be the required velocity of emergence, we have

$$v^2=(200)^2-2f\,.\,\frac{5}{12}=(200)^2\left(1-\frac{5}{9}\right), \text{ so that } v=133\tfrac{1}{3} \text{ feet per second.}$$

25. The acceleration $=v\div t=\dfrac{40\times 10^5}{60\times 60}\div 4=\dfrac{10^4}{36}$, and therefore

$$s=\tfrac{1}{2}ft^2=\frac{1}{2}\,.\,\frac{10^4}{36}\times 16 \text{ cms.}=22\tfrac{2}{9} \text{ metres.}$$

Also $$\frac{\text{Force}}{\text{Weight}}=\frac{f}{g}=\frac{10^4}{36\times 981}=\frac{2500}{8829}=\cdot 283\ldots,$$

and if the incline be 1 in x, so that $\sin\alpha=\dfrac{1}{x}$, then

$$\frac{1}{x}=\frac{f}{g}=\frac{10^4}{36\times 981}, \text{ i.e. } x=3\cdot 53\ldots.$$

Page 71, Art. 70.

Ex. 2. What appears to be 3224 lbs. at A only appears to weigh 3212 lbs. at B. Hence, since what costs £10 must sell for £12, the selling price

$$=\frac{3224}{3212}\times £12=£12.\ 0s.\ 10\tfrac{3}{4}d.$$

EXAMPLES. X. (Pages 81, 82.)

1. Let T poundals be the tension of the string, and f be the magnitude of the common acceleration. Then we have

$$9f=9g-T \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$6f=T-6g \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), by addition, we have $f=\dfrac{g}{5}$.

Also, from (2),

$$T=6(f+g)=g\left(6+\frac{6}{5}\right)=\frac{36}{5}g \text{ poundals}=7\tfrac{1}{5} \text{ lbs wt.}$$

Otherwise, by direct substitution in the formulæ

$$f=\frac{m_1-m_2}{m_1+m_2}g, \text{ and } T=\frac{2m_1m_2}{m_1+m_2}g.$$

2 Here (1) $f = \dfrac{9-7}{9+7} g = \dfrac{2}{16} g = 4$ ft.-sec. units,

(2) $T = \dfrac{2 \cdot 7 \cdot 9}{7+9} g = \dfrac{126}{16}$ lbs. wt. $= 7\frac{7}{8}$ lbs. wt.,

(3) The velocity $= ft = 4 \times 5 = 20$ feet per second;

and (4) The distance described $= \dfrac{1}{2} \cdot 4 \cdot 5^2 = 50$ feet.

3. If T poundals be the tension of the string, and f be the common acceleration, we have

$$13g - T = 13f, \text{ and } T - 11g = 11f.$$

Hence, by addition $f = \dfrac{g}{12}$. Thus

(1) The common velocity at the end of 4 seconds

$$= ft = \frac{g}{12} \times 4 = \frac{4 \times 32}{12} = 10\tfrac{2}{3} \text{ feet per second.}$$

(2) The distance described in 4 seconds

$$= \frac{1}{2} \cdot \frac{g}{12} \cdot 4^2 = 8 \times \frac{32}{12} = 21\tfrac{1}{3} \text{ feet.}$$

If the string be now cut the distance described downwards by the larger mass in the next 6 seconds

$$= 10\tfrac{2}{3} \times 6 + \frac{1}{2} \times g \times 6^2 = 640 \text{ feet;}$$

and the distance described upwards by the smaller mass

$$= 10\tfrac{2}{3} \times 6 - \frac{1}{2} \times g \times 6^2 = -512 \text{ feet,}$$

i.e. 512 feet *downwards*.

4. If f be the common acceleration, we have

$$f = \frac{550-450}{550+450} g = \frac{g}{10} = \frac{981}{10};$$

hence the masses go in the first 3 seconds of the motion distances

$$= \frac{1}{2} \cdot \frac{981}{10} \cdot 3^2 \text{ cms.} = 4{\cdot}41 \text{ metres.}$$

Also the tension of the string

$$= \frac{2m_1 m_2}{m_1 + m_2} g \text{ dynes} = \frac{2 \cdot 450 \cdot 550}{450 + 550} g \text{ dynes}$$

$$= 495 \text{ grammes' wt.}$$

5. Let the tension of the cord be T poundals. Then

$$T = \frac{2 \cdot 5 \cdot 7}{5+7} \text{ lbs. wt.} = \frac{35}{6} \text{ lbs. wt.};$$

hence the pull on the hook

$$= 2T = \frac{35}{3} \text{ lbs. wt.} = 11\tfrac{2}{3} \text{ lbs. wt.}$$

6. The pressure originally was $(3+3)$ lbs. wt., *i.e.* 6 lbs. wt. When the additional mass of 3 lbs. is placed on one of the masses, the tension of the string

$$= \frac{2 \cdot 3 \cdot 6}{3+6} \, g \text{ poundals} = 4 \text{ lbs. wt.}$$

Hence the new pressure $= 2T = 8$ lbs. wt., and the increase is therefore

$(8-6)$ lbs. wt., *i.e.* 2 lbs. wt.

7. The pressure originally was

$(P+P)$ lbs. wt., *i.e.* $2P$ lbs. wt.

When the additional weight P is placed on one of the weights, the tension of the string

$$= \frac{2 \cdot P \cdot 2P}{P+2P} \, g \text{ poundals} = \frac{4P}{3} \text{ lbs. wt.}$$

Hence the new pressure

$$= 2T = \frac{8P}{3} \text{ lbs. wt.,}$$

and the increase is therefore

$$\left(\frac{8P}{3} - 2P\right) \text{ lbs. wt., i.e. } \frac{2P}{3} \text{ lbs. wt.}$$

8. Let x be the required mass, so that the masses are now $m+x$ and $m-x$. The acceleration

$$= \frac{(m+x)-(m-x)}{(m+x)+(m-x)} \, g = \frac{x}{m} \, g. \quad \text{(Art. 74.)}$$

Hence $$200 = \frac{1}{2} \cdot \frac{x}{m} \cdot g \cdot 5^2 = 400 \, \frac{x}{m}, \text{ so that } x = \frac{m}{2}.$$

9. The acceleration of the system

$$= \frac{3-2}{3+2} \cdot g = \frac{g}{5};$$

therefore after 5 seconds the velocity of the system $= g$. Hence, if h be the height afterwards ascended by the mass of 2 lbs., we have

$$g^2 = 2gh, \text{ i.e. } h = \frac{g}{2} = 16 \text{ feet.}$$

10. (1) By Art. 75, the common acceleration

$$=\frac{1}{9+1}g=\frac{g}{10}.$$

(2) The required time t is given by

$$8=\frac{1}{2}\cdot\frac{g}{10}\cdot t^2, \text{ so that } t=\sqrt{5} \text{ seconds.}$$

(3) The required velocity

$$=ft=\frac{g}{10}\sqrt{5}=\frac{16}{5}\sqrt{5} \text{ feet per second.}$$

11. The acceleration of the system

$$=\frac{50}{350+50}g=\frac{g}{8}=\frac{981}{8};$$

hence, if t seconds be the required time, we have

$$245\cdot 25=\frac{1}{2}\cdot\frac{981}{8}t^2, \text{ whence } t=2 \text{ seconds.}$$

12. By Art. 76,

$$f=\frac{3-5\sin 30^\circ}{3+5}g=2 \text{ ft.-sec. units,}$$

and $$T=\frac{3.5(1+\sin 30^\circ)}{3+5}g \text{ poundals}=\frac{45}{16} \text{ lbs. wt.}=2\tfrac{13}{16} \text{ lbs. wt.}$$

Also the required velocity $=ft=2\times 3=6$ feet per second, and the required space

$$=\frac{1}{2}.2.3^2=9 \text{ feet.}$$

13. The common acceleration

$$=\frac{3-4\sin 45^\circ}{3+4}g=\frac{3-2\sqrt{2}}{7}g. \qquad \text{(Art. 76.)}$$

Hence, if t seconds be the required time, we have

$$7=\frac{1}{2}\cdot\frac{3-2\sqrt{2}}{7}g.t^2,$$

i.e. $$t^2=\frac{7^2}{16}\cdot\frac{3+2\sqrt{2}}{(3+2\sqrt{2})(3-2\sqrt{2})}=\frac{7^2}{16}\cdot\frac{3+2\sqrt{2}}{9-8}.$$

$$\therefore\ t=\frac{7}{4}\sqrt{3+2\sqrt{2}}=\frac{7}{4}(\sqrt{2}+1) \text{ seconds.}$$

14. The inclination of the plane to the horizon

$$=\alpha=\sin^{-1}\frac{1}{2},$$

so that the acceleration

$$=\frac{8-12\sin\alpha}{8+12}g=\frac{8-6}{20}g=\frac{16}{5},$$

and the required distance

$$=\frac{1}{2}\cdot\frac{16}{5}\cdot 5^2=40 \text{ feet}.$$

15. We have

$$\frac{6}{16}g\sin\alpha-T=\frac{6}{16}f,\ T=mf,\ \text{and}\ 3=\frac{1}{2}f\cdot 5^2,$$

$$\therefore\ \frac{3}{8}g\sin\alpha=\left(\frac{3}{8}+m\right)f=\left(\frac{3}{8}+m\right)\frac{6}{25}.$$

$$\therefore\ m=\frac{25}{6}\times\frac{3}{8}\cdot\frac{g}{2}-\frac{3}{8}=\left(25-\frac{3}{8}\right)\text{lbs.}=24 \text{ lbs. } 10 \text{ oz.}$$

16. Let the mass be x oz., so that

$$f=\frac{x-4}{x+4}\cdot g.$$

The velocity at the end of 3 seconds is $3f$; hence we have

$$0=(3f)^2-2\cdot g\cdot\frac{16}{9},\ \text{so that}\ f=\frac{32}{9}.$$

Hence $$\frac{x-4}{x+4}=\frac{1}{9},\ i.e.\ x=5 \text{ oz}.$$

17. Let x lbs. be put into one scalepan, and $(12-x)$ lbs. into the other, so that the masses on the two sides of the pulley are now $(x+3)$ lbs. and $(15-x)$ lbs.

Then $$f=\frac{2x-12}{18}g=\frac{x-6}{9}g.$$

But $$50=\frac{1}{2}f\cdot 5^2,\ \text{so that}\ f=4.$$

$$\therefore\ \frac{(x-6)}{9}g=4,\ i.e.\ x=\frac{57}{8}.$$

Hence the required ratio $=\dfrac{x}{12-x}=\dfrac{57}{96-57}=\dfrac{57}{39}=\dfrac{19}{13}$.

18. Let T_1 and T_2 poundals be the tensions of the strings, and f be the common acceleration. Then

$$3g-T_1=3f,\ 4g-T_2=4f,\ \text{and}\ T_1+T_2-5g=5f.$$

Hence, by addition, $$f=\frac{g}{6}.$$

Also, by substitution,

$$T_1=2\tfrac{1}{2}g=2\tfrac{1}{2} \text{ lbs. wt.},$$

and $$T_2=3\tfrac{1}{3}g=3\tfrac{1}{3} \text{ lbs. wt.}$$

19. The original acceleration

$$=\frac{8+4-10}{8+4+10}\cdot g=\frac{g}{11},$$

and the final acceleration

$$=\frac{10-8}{10+8}\cdot g=\frac{g}{9}$$

in the opposite direction. After 5 seconds the velocity is $\frac{5g}{11}$, and hence the required distance is given by the equation

$$\left(\frac{5g}{11}\right)^2=2\cdot\frac{g}{9}\cdot x,$$

i.e. $$x=\frac{225}{242}\cdot g \text{ feet}=\text{about } 29\tfrac{3}{4} \text{ feet.}$$

20. If m_1 and m_2 be the masses, and T poundals be the tension of the string, then the pressure on the axis

$$=2T=2\times\frac{2m_1m_2}{m_1+m_2}g=2\times\frac{2w_1w_2}{w_1+w_2},$$

if $w_1=m_1g$ and $w_2=m_2g$; hence the pressure

$$=\frac{(w_1+w_2)^2-(w_1-w_2)^2}{w_1+w_2}=w_1+w_2-\frac{(w_1-w_2)^2}{w_1+w_2},$$

which is less than w_1+w_2.

21. The moving force is equal to the difference of the weights Pg and Qg, *i.e.* to $(P-Q)g$, if $P>Q$. The mass moved is equal to $P+Q+M$. Hence the required acceleration

$$=\frac{P-Q}{P+Q+M}g.$$

Otherwise thus: If f be the required acceleration, $(P>Q)$ and if T and T_1 be the tensions of the strings joining P and M, and M and Q, respectively, we have

$$Pf=Pg-T,\quad Mf=T-T_1,\quad \text{and}\quad Qf=T_1-Qg.$$

Hence, by addition, we have

$$(P+M+Q)f=(P-Q)g,$$

so that $$f=\frac{P-Q}{P+Q+M}\cdot g.$$

EXAMPLES. XI. (Pages 86—88.)

1. Let f be the acceleration, and T poundals be the tension of the string. The friction is 5μ lbs. wt.; for the motion of the mass of 8 lbs. we have $8f=8g-T$; and for the motion of the mass of 5 lbs., $5f=T-5\mu g$; therefore, by addition, $13f=8g-5\mu g$. Hence

$$(1)\ \text{if}\ \mu=\frac{1}{2},\ \text{we have}\ f=\frac{8-2\frac{1}{2}}{13}g=\frac{11}{26}g;$$

and

$$(2)\ \text{if}\ f=\frac{g}{2},\ \text{we have}\ \mu=\frac{8-6\frac{1}{2}}{5}=\frac{3}{10}=\cdot 3.$$

2. The original acceleration

$$=\frac{3Q-Q\sqrt{3}}{3Q+Q}g=8(3-\sqrt{3}).$$

The distance described in 4 seconds

$$=\frac{1}{2}\cdot 8(3-\sqrt{3})\times 4^2=64(3-\sqrt{3})\text{ feet.}$$

The velocity acquired $=32(3-\sqrt{3})$. Also the new acceleration

$$=-\sqrt{3}g.$$

The distance described before the mass Q is at rest is, therefore,

$$\frac{[32(3-\sqrt{3})]^2}{2\sqrt{3}g},\ i.e.\ 32(2\sqrt{3}-3)\text{ feet.}$$

Hence the required distance

$$=64(3-\sqrt{3})+32(2\sqrt{3}-3)=96\text{ feet.}$$

3. If f be the acceleration and μ be the coefficient of friction, we have $40f=40g-T$, and $200f=T-200\mu g$; therefore, by addition,

$$24f=(4-20\mu)g.$$

Also the acceleration would be doubled if the table were smooth,

$$i.e.\ \text{if}\ \mu=0;\ \text{hence}\qquad \frac{1-5\mu}{6}g=f=\frac{1}{2}\cdot\frac{g}{6},$$

$$i.e.\qquad 1=2(1-5\mu),\ \text{whence}\ \mu=\frac{1}{10}=\cdot 1.$$

4. If R be the normal reaction of the plane in poundals,

$$R=10g\cos 30^\circ=5g\sqrt{3};$$

hence, resolving up the plane, f being the upward acceleration of the mass of 10 lbs., we have

$$10f=15g-10g\sin 30^\circ-\mu\cdot 5g\sqrt{3}=15g-5g-5g=5g,$$

so that $f=16$; hence, if t and v be the required time and velocity,

$$4=\frac{1}{2}ft^2=8t^2, \text{ whence } t=\frac{\sqrt{2}}{2} \text{ second,}$$

and $$v=ft=16\times\frac{\sqrt{2}}{2}=8\sqrt{2} \text{ feet per second.}$$

5. In this case the force of 15 lbs. wt. is replaced by T the tension of the string, and for the motion of the hanging mass we have

$$15f=15g-T;$$

hence $$10f=15(g-f)-5g-5g=5g-15f,$$

$$\therefore 25f=5g, \text{ and } f=\frac{g}{5},$$

$$\therefore 4=\frac{1}{2}ft^2=\frac{gt^2}{10},$$

and hence $$t=\sqrt{\frac{40}{32}}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2} \text{ second;}$$

also $$v=ft=\frac{g}{5}\cdot\frac{\sqrt{5}}{2}=\frac{16}{5}\sqrt{5} \text{ feet per second.}$$

6. If $$\alpha=\sin^{-1}\frac{3}{5},$$

the acceleration f down the plane

$$=g\sin\alpha-\mu g\cos\alpha=g\left(\frac{3}{5}-\frac{1}{2}\cdot\frac{4}{5}\right)=\frac{g}{5};$$

hence the body slides down in time t given by

$$100=\frac{1}{2}\cdot\frac{g}{5}\cdot t^2,$$

i.e. $$t^2=\frac{1000}{32}=\frac{125}{4}, \text{ and } t=\frac{5\sqrt{5}}{2} \text{ seconds.}$$

Also the velocity at the lowest point

$$=ft=\frac{g}{5}\cdot\frac{5\sqrt{5}}{2}=16\sqrt{5} \text{ feet per second.}$$

If the body were projected up the plane with velocity u, the retardation would be

$$g\sin\alpha+\mu g\cos\alpha, \textit{ i.e. } g\left(\frac{3}{5}+\frac{1}{2}\cdot\frac{4}{5}\right), \textit{ i.e. } g,$$

so that $$u^2=2g\times100=64\times100,$$

and $$u=80 \text{ feet per second.}$$

7. The acceleration down the rough plane

$$=g\left(\sin\frac{\pi}{4}-\frac{3}{4}\cos\frac{\pi}{4}\right)=\frac{g}{4\sqrt{2}};$$

also the acceleration down the smooth plane

$$=g\sin\frac{\pi}{4}=\frac{g}{\sqrt{2}};$$

hence, using the formula $\quad s=\frac{1}{2}ft^2,$

if t and t' be the respective times, we have

$$t=\sqrt{2s\div\frac{g}{4\sqrt{2}}},\ \text{and}\ t'=\sqrt{2s\div\frac{g}{\sqrt{2}}};$$

hence $$t=2t'.$$

8. If f be the acceleration, T be the tension of the string, and R and S be the normal reactions of the planes respectively, we have

$$R=5g\cos 30°,\ \text{and}\ S=10g\cos 60°;$$

also, for the motions of the masses of 5 lbs. and 10 lbs. respectively,

$$5f=T-5g(\sin 30°+\mu\cos 30°),$$

and $$10f=10g(\sin 60°-\mu\cos 60°)-T.$$

Hence, by addition,

$$15f=10g\left(\frac{\sqrt{3}}{2}-\frac{1}{2\sqrt{3}}\right)-5g\left(\frac{1}{2}+\frac{1}{2}\right),$$

so that $$f=\frac{2\sqrt{3}-3}{9}g.$$

The direction of motion is down the more elevated plane, as the mass on the other plane cannot slide down it, since $\mu=\tan 30°$.

9. Since the mass of 10 lbs. cannot slide down the plane of inclination 30°, the motion, if any, is in the same direction as before, and with the same notation, for the motion of the two masses, we have

$$10f=T-10g(\sin 30°+\mu\cos 30°)=T-10g,$$

and $$5f=5g(\sin 60°-\mu\cos 60°)-T;$$

$$\therefore\ 15f=5g\left(\frac{\sqrt{3}}{2}-\frac{1}{2\sqrt{3}}\right)-10g,$$

whence $$f=\frac{g}{6\sqrt{3}}(5-4\sqrt{3}),$$

which is negative; this shews that the motion in the direction supposed is impossible, and therefore there can be no motion at all. It might be supposed that f equal to a negative quantity implied motion in the opposite direction, but for such motion the equations have to be altered, and the result does not hold. This is an example of the discontinuity which often arises in problems with friction.

10. 15 miles per hour $=22$ feet per second.

If the train be a mass of m lbs., and f be the retardation, the equation $P=mf$ gives $\frac{8 \cdot mg}{2240}=mf$, whence $f=\frac{4}{35}$ ft.-sec. units; the formula

$$v^2=2fs \text{ gives } (22)^2=2 \cdot \frac{4}{35} \cdot s,$$

whence the required distance $=s=2117\frac{1}{2}$ feet.

11. 30 miles per hour $=44$ feet per second.

We have $$(44)^2=2f \cdot 60 \cdot 3,$$

whence $$f=\frac{242}{45} \text{ ft.-sec. units;}$$

hence, if the friction be x lbs. per ton, and m be the mass of the train,

$$mf=\frac{mxg}{2240} \text{ poundals.}$$

$$\therefore\ x : 2240=f : g=\frac{242}{45} : 32=121 : 720.$$

Hence the whole friction on the train

$$=\frac{200 \times 121}{720} \text{ tons wt.}=\frac{605}{18} \text{ tons wt.;}$$

i e. the whole friction : 1 ton wt. $=605 : 18$.

12. Here the velocity $=44$ feet per second, and the retardation

$$=\frac{10g}{2240}=\frac{1}{7}.$$

Hence (1) the time t is such that

$$44=\frac{1}{7}t,$$

i.e. $t=308$ seconds $=5$ minutes 8 seconds;

and (2) the distance s is such that

$$(44)^2=2 \cdot \frac{1}{7} \cdot s, \text{ whence } s=6776 \text{ feet.}$$

13. Here the retardation is increased by $g \sin \alpha$, *i.e.* by $g \cdot \frac{1}{112}$, *i.e.* by $\frac{2}{7}$, and therefore is equal to $\frac{3}{7}$. Hence we have

$$44=\frac{3}{7}t,$$

so that $t=102\frac{2}{3}$ seconds $=1$ minute $42\frac{2}{3}$ seconds;

also $$(44)^2=\frac{6}{7}s, \text{ whence } s=2258\frac{2}{3} \text{ feet.}$$

14. $$40 \text{ miles per hour} = \frac{176}{3} \text{ feet per second.}$$

Let f be the retardation caused by the resistance to the motion, so that

$$\left(\frac{176}{3}\right)^2 = 2\left(f - \frac{g}{120}\right) \times 880 \times 3;$$

hence $$f = \frac{g}{120} + \frac{88}{135}.$$

The required resistance therefore

$$= \text{wt. of } 200\left(\frac{1}{120} + \frac{88}{32 \times 135}\right) \text{tons} = 5\tfrac{20}{27} \text{ tons wt.}$$

15. The acceleration down the incline

$$= \frac{g}{100} - \frac{8g}{2240} = \frac{9g}{1400}.$$

The velocity v at the bottom of the incline is therefore given by

$$v^2 = 2 \cdot \frac{9g}{1400} \times 1760 \times 3.$$

On the level the acceleration is $-\frac{g}{280}$. Hence, assuming the velocity to be unaltered as the train passes from the incline to the level, the train will be at rest after describing a distance x, where

$$v^2 = 2 \cdot \frac{g}{280} \cdot x.$$

Equating these two values of v, we have

$$x = (27 \times 352) \text{ feet} = 3168 \text{ yards} = 1 \text{ mile } 1408 \text{ yards.}$$

16. The incline is at an angle α to the horizon, where

$$\sin\alpha = \frac{1}{128}.$$

The moving force down the incline = the resolved part of the weight − the resistance due to friction, etc.

$$= \left(140 \times \frac{1}{128} - 140 \times \frac{10}{2240}\right) \text{tons wt.}$$

$$= 140\left(\frac{1}{128} - \frac{1}{224}\right) = 140 \times \frac{3}{896} \text{ tons wt.};$$

therefore the acceleration down the incline $= \frac{3}{896} \cdot g = \frac{3}{28}$.

The velocity at the top of the incline is 15 miles per hour, *i.e.* 22 feet per second; hence the velocity v at the foot of the incline is given by

$$v^2=u^2+2fs=(22)^2+2\cdot\frac{3}{28}\cdot 880\cdot 3$$

$$=(22)^2+\frac{9\times 440}{7}=44\left(11+\frac{90}{7}\right).$$

Along the horizontal line the retarding force

$$=140\times\frac{10}{2240}=140\times\frac{1}{224}\text{ tons wt.};$$

so that the retardation $\quad =\dfrac{1}{224}\cdot g=\dfrac{1}{7}.$

Hence, if x feet be the required distance, the equation

$$v^2=u^2-2fx \text{ gives } 0=44\left(11+\frac{90}{7}\right)-\frac{2}{7}x,$$

so that $\quad x=\dfrac{7}{2}\times 44\left(11+\dfrac{90}{7}\right)=3674 \text{ feet}=1224\tfrac{2}{3}\text{ yards}.$

17. Here the retarding force along the horizontal line

$$=\left(\frac{1}{224}\times 130+\cdot 5\times 10\right)\text{ tons wt.}=\frac{625}{112}\text{ tons wt.},$$

so that the retardation $\quad =\dfrac{625}{112}\times\dfrac{g}{140}=\dfrac{125}{98}.$

Also the velocity at the foot of the incline

$$=\sqrt{44\left(11+\frac{90}{7}\right)}=\sqrt{\frac{7348}{7}},$$

as in the last example. Hence the required distance x feet is given by

$$0=\frac{7348}{7}-\frac{250}{98}x,\text{ so that } x=411\tfrac{61}{125}\text{ feet.}$$

18. $\quad$ 45 miles per hour $=$ 66 feet per second;

thus we have $\quad (66)^2=2\cdot f\cdot 1760\cdot 3$, so that $f=\dfrac{33}{80}.$

Let F tons wt. be the force exerted by the engine; then, since the resistance

$$=\frac{1}{50}(130+30)=\frac{16}{5}\text{ tons,}$$

we have, by the equation $P=mf$,

$$F-\frac{16}{5}=\frac{160}{g}\cdot\frac{33}{80}=2\tfrac{1}{16}.$$

$$\therefore\ F=2\tfrac{1}{16}+3\tfrac{1}{5}=5\tfrac{21}{80}\text{ tons wt.}$$

Again, if the incline be at an angle a to the horizon, then if $F_{,,}$ were just equal to

$$\frac{16}{5}+160\sin a,$$

the train would not move, and this gives

$$\sin a=\frac{5\frac{21}{80}-3\frac{1}{5}}{160}=\frac{1}{77}, \text{ nearly};$$

i.e. the incline rises 1 in 77 nearly.

EXAMPLES. XII. (Pages 93—97.)

1. There is no pressure, since the weight and the hand are both descending with the same acceleration g.

Otherwise thus: Let R denote the pressure; then since the system is moving downwards with acceleration g, we have

$$20g=20g-R, \text{ i.e. } R=0.$$

2. (1) The velocity being constant, the acceleration of the plane is zero; hence, if R poundals be the pressure on the mass upwards,

$$R-20g=0.$$

$$\therefore\ R=20g \text{ poundals}=20 \text{ lbs. wt.}$$

(2) $$R-20g=20\times 1, \text{ so that } R=20\tfrac{5}{8} \text{ lbs. wt.}$$

3. (1) If P be the required pressure, then

$$P-112g=112\times 12;$$

hence $$P=112(g+12)=112\left(1+\frac{3}{8}\right) \text{ lbs. wt.}=154 \text{ lbs. wt.}$$

(2) In this case $$112g-P_1=112\times 12,$$

so that $$P_1=112\left(1-\frac{3}{8}\right) \text{ lbs. wt.}=70 \text{ lbs. wt.}$$

4. The resultant vertical force on the coal

$$=(126-112) \text{ lbs. wt.}=14 \text{ lbs. wt.}=14g.$$

Hence the acceleration $$=\frac{14g}{112}=\frac{g}{8}.$$

5. If f be the acceleration of the balloon, then

$$116g-112g=112f, \text{ so that } f=\frac{g}{28}=\frac{8}{7}.$$

Hence the required height

$$=\frac{1}{2}f\,.\,(60)^2=2057\tfrac{1}{7} \text{ feet.}$$

6. On one side the total mass is 330 grammes, and on the other side is 270 grammes. The acceleration f of the system is therefore

$$\frac{330-270}{330+270}g, \text{ i.e. } \frac{g}{10}.$$

Also the tension of the string

$$=\frac{2m_1m_2}{m_1+m_2}\cdot g=\frac{2\,.\,330\,.\,270}{330+270}\cdot g=297g \text{ dynes}=297 \text{ grammes' wt.}$$

If P_1 and P_2 be the pressures on the scale-pans respectively, we have

$$300g-P_1=300f, \text{ and } P_2-240g=240f.$$

$$\therefore\ P_1=270g=270 \text{ grammes' wt., and } P_2=264g=264 \text{ grammes' wt.}$$

7. On one side the total mass is 3 oz and on the other side is 5 oz. The acceleration f of the system is therefore

$$\frac{5-3}{5+3}g, \text{ i.e. } \frac{g}{4}.$$

Also the tension of the string

$$=\frac{2m_1m_2}{m_1+m_2}\cdot g=\frac{2\cdot\frac{5}{16}\cdot\frac{3}{16}}{\frac{5}{16}+\frac{3}{16}}g=\frac{15}{64}g \text{ poundals}=3\tfrac{3}{4} \text{ oz. wt.}$$

If P_1 and P_2 be the pressures on the scale-pans respectively, we have

$$P_1-\frac{2}{16}g=\frac{2}{16}f, \text{ and } \frac{4}{16}g-P_2=\frac{4}{16}f,$$

whence $$P_1=\frac{1}{8}\left(g+\frac{g}{4}\right)=\frac{5g}{32} \text{ poundals}=2\tfrac{1}{2} \text{ oz. wt.,}$$

and $$P_2=\frac{1}{4}\left(g-\frac{g}{4}\right)=\frac{3g}{16} \text{ poundals}=3 \text{ oz. wt.}$$

8. On each square foot there fell per second

$$\left(\frac{1}{2}\cdot\frac{1}{12}\cdot\frac{1}{3\,.\,60\,.\,60}\right) \text{ cubic feet, i.e. } \frac{10}{32\,.\,36\,.\,36} \text{ lbs.}$$

Hence the impulsive pressure per square foot

$$=\frac{10}{32\,.\,36\,.\,36}\cdot 10 \text{ poundals.}$$

The impulsive pressure in tons wt. per square mile therefore

$$=\left(\frac{100}{32.36.36}\times 9\times 4840\times 640\right)\div 32\times 2240=\cdot 938,\text{ nearly.}$$

9. On each square foot there fell per second

$$\left(\frac{1}{4}\cdot\frac{1}{24.60.60}\right)\text{ cubic feet, } i.e. \frac{5}{96.36.8}\text{ lbs.}$$

Also the final velocity

$$=\sqrt{2g.400}=160\text{ feet per second;}$$

hence the pressure per square foot

$$=\frac{5\times 160}{96\times 36\times 8}=\frac{100}{96\times 36}\text{ poundals,}$$

so that the pressure per acre

$$=9\times 4840\times\frac{100}{96\times 36}=1260\tfrac{5}{12}\text{ poundals.}$$

10. The momentum destroyed by the wall per second

$$=\left(80\times\frac{300\times 277\frac{1}{4}}{1728}\times\frac{1000}{16}\right)\text{ lbs.}=\frac{1500000\times 1109}{4\times 1728}\text{ lbs.;}$$

hence the required pressure

$$=\frac{1500000\times 1109}{32\times 4\times 1728}\text{ lbs. wt.}=7521\text{ lbs. wt., nearly.}$$

11. If f be the acceleration, we have

$$10\text{ metres}=1000\text{ centimetres}=\frac{1}{2}.f.100,$$

whence $f=20$ C.G.S. units; but

$$f=\frac{240+10-240}{240+10+240}.g=\frac{g}{49},$$

so that $$g=49f=49\times 20=980.$$

12. Take the notation of Art. 82. So long as P and Q are constant we have to prove that f is constant.

But $$T=\frac{h_1}{\sqrt{2fh}},$$

so that $$f=\frac{1}{2}\frac{h_1^2}{hT^2}.$$

We have therefore to shew, for different positions of the platform E, that $\frac{h_1^2}{hT^2}$ is constant, so long as P and Q remain the same.

13. Suppose that originally x balls hung vertically. We then have

$$\frac{g}{2}=\frac{x-(16-x)\sin a}{x+(16-x)}g=\frac{4x-16}{48}g=\frac{x-4}{12}g.$$

Hence $$x=10.$$

14. The acceleration f

$$=\frac{P-Q\sin 30^\circ}{P+Q}g=\frac{2P-Q}{P+Q}\cdot\frac{g}{2}.$$

Since $$s=\frac{1}{2}ft^2,$$

the time of describing s is equal to $\sqrt{\frac{2s}{f}}$, and the time of falling freely is equal to $\sqrt{\frac{2s}{g}}$; therefore, by hypothesis,

$$\sqrt{\frac{2s}{f}}=4\sqrt{\frac{2s}{g}}, \text{ whence } f=\frac{g}{16}.$$

Thus we have $$\frac{2P-Q}{P+Q}\cdot\frac{g}{2}=\frac{g}{16};$$

hence $$P:Q=3:5.$$

15. When P drags Q, the acceleration

$$=\frac{9-6\sin 30^\circ}{9+6}\cdot g=\frac{2}{5}g.$$

When Q drags P, the acceleration

$$=\frac{6-9\sin 30^\circ}{6+9}\cdot g=\frac{g}{10}.$$

Hence, if t_1 and t_2 be the respective times, we have

$$\frac{1}{2}\cdot\frac{2g}{5}\cdot t_1^2=\frac{1}{2}\cdot\frac{g}{10}\cdot t_2^2, \text{ so that } t_2=2t_1.$$

16. The length of the inclined plane $=20$ feet; therefore if a be the inclination of the plane to the horizon,

$$\sin a=\frac{12}{20}=\frac{3}{5}, \text{ and } \cos a=\frac{4}{5}.$$

The acceleration down the plane

$$=g\sin a-\frac{1}{8}g\cos a=g\left(\frac{3}{5}-\frac{1}{8}\cdot\frac{4}{5}\right)=\frac{g}{2},$$

so that, if v be the acquired velocity at the foot of the inclined plane,

$$v^2=2\cdot\frac{g}{2}\cdot 20=20g.$$

Hence, if x feet be the required distance along the horizontal plane, the retardation being $\frac{1}{8}.g$, we have

$$20g=2.\frac{g}{8}.x, \text{ and therefore } x=80 \text{ feet.}$$

17. If mg be the weight of the train, the retarding force

$$=\cdot16\times\frac{3}{4}\times mg=mf,$$

so that
$$f=\frac{16}{100}\times\frac{3}{4}\times32=\frac{96}{25}.$$

Hence, since 30 miles per hour $=44$ feet per second, the train will be brought to rest after traversing a distance s feet given by

$$(44)^2=2.\frac{96}{25}.s.$$

$$\therefore s=252 \text{ feet, nearly}=84 \text{ yards, nearly.}$$

18. 30 miles per hour $=44$ feet per second.

Since a velocity of 44 feet per second is destroyed by an acceleration f in 1320 feet, we have

$$f=\frac{(44)^2}{2\times1320}=\frac{11}{15}.$$

Hence, if m be the mass of the brake-van in tons, we have

$$50\times\frac{11}{15}=\mu\times mg=\frac{g}{6}.m, \text{ i.e. } m=\frac{110}{3}\times\frac{6}{32}=6\tfrac{7}{8} \text{ tons}$$

19. The acceleration f at first

$$=\frac{m'-m\sin\alpha}{m'+m}.g=\frac{lm'-hm}{m+m'}.\frac{g}{l}.$$

If m' be detached when a distance x has been described, we have

$$v^2=2fx.$$

With the new acceleration $-g\frac{h}{l}$ the velocity v is destroyed in a distance $l-x$, so that

$$v^2=2.g\frac{h}{l}(l-x).$$

Equating these values of v, we have

$$x=\frac{ghl}{lf+gh}=\frac{m+m'}{m'}.\frac{hl}{h+l}.$$

20. Let m_1 and m_2 be the masses, $m_1 > m_2$, T be the tension, and $m_1 + m_2 = M$. We have

$$f = \frac{m_1 - m_2}{m_1 + m_2} g, \text{ and } T = \frac{2m_1 m_2}{m_1 + m_2} g.$$

$$\therefore\ T = \frac{1}{2} \cdot \frac{(m_1 + m_2)^2 - (m_1 - m_2)^2}{m_1 + m_2} g$$

$$= \frac{1}{2} \left[(m_1 + m_2) g - \frac{(m_1 - m_2)^2}{m_1 + m_2} g \right]$$

$$= \frac{M}{2} \left(g - \frac{f^2}{g} \right), \text{ i.e. } T \text{ is greater the less } f \text{ is.}$$

21. We have $\quad \dfrac{2m_1 m_2}{m_1 + 2m_2} g \cdot \dfrac{m_1 m_2}{m_1 + m_2} g = 3 : 2,$

whence $\quad m_1 = 2m_2$, *i.e.* $m_1 \quad m_2 = 2 : 1$.

22. The accelerations are $\frac{g}{9}$ and $\frac{g}{16}$ respectively; the distances moved in time t are

$$\frac{1}{2} \cdot \frac{g}{9} \cdot t^2 \text{ and } \frac{1}{2} \cdot \frac{g}{16} \cdot t^2;$$

hence $$\frac{16t^2}{9} + t^2 = 10,$$

i.e. $$t = \frac{3}{5}\sqrt{10} \text{ second} = 1{\cdot}9 \text{ second, nearly.}$$

23. Since $P > \frac{W}{2}$, P descends Let f be the acceleration of P downwards; then $\frac{f}{2}$ is the acceleration of W upwards. The equations of motion are

$$P - T = \frac{P}{g} \cdot f, \text{ and } 2T - W = \frac{W}{g} \cdot \frac{f}{2},$$

$$\therefore\ \frac{P - T}{P} = \frac{2T - W}{\frac{1}{2} W}, \text{ whence } T = \frac{3PW}{W + 4P}.$$

24. By Statics (third system of pulleys), originally

$$M = (2^2 - 1) m = 3m$$

Since $M + m$ moves through one-third of the distance that m does, let f and $3f$ be the accelerations respectively. For m, we have

$$T_1 - mg = m \cdot 3f;$$

and for $M + m$, we have

$$(M + m) g - T_1 - T_2 = (M + m) f,$$

[T_1 and T_2 being as in the figure p. 190, Statics, where also $P = m$ and $W = M$]; also $T_2 = 2T_1$.

Hence the equations are

$$T_1=m(3f+g), \text{ and } 3T_1=4m(g-f);$$

whence $$f=\frac{g}{13}, \text{ and therefore } 3f=\frac{3g}{13}.$$

25. [Cf. the figure p. 182, Statics, with three movable pulleys, and string T_3 passing over a fixed pulley.] If the body rise a height x, so does the lowest pulley; the middle pulley rises $2x$, and the highest pulley $4x$; therefore the length of the vertical string attached to the body of mass 15 lbs. is increased by $8x$, *i.e.* the power descends through a distance $8x$. Hence if the body move with acceleration f, the acceleration of the power is $8f$; and we have for the body of mass 1 cwt., $112(f+g)=2T_1$; for the body of mass 15 lbs., $15 \cdot 8f=15g-T_3$; also for the middle pulley, $2T_2=T_1$, and for the upper pulley, $2T_3=T_2$. Thus we have

$$T_2=\frac{56}{2}(f+g), \text{ and } T_3=\frac{28}{2}(f+g).$$

$$\therefore 14(f+g)=15(g-8f), \text{ so that } f=\frac{g}{134}.$$

26. The tension T is the same throughout, and since each pulley M is at rest, $2T=Mg$; also the pulleys being smooth, the string will slide round them with acceleration f, where

$$mf=mg-T, \text{ and } m'f=T-m'g,$$

so that $$f=\frac{m-m'}{m+m'}\cdot g,$$

and $$T=mg\left(1-\frac{m-m'}{m+m'}\right)=mg\cdot\frac{2m'}{m+m'}.$$

Hence the required condition is that

$$Mg=\frac{4mm'}{m+m'}\cdot g, \text{ i.e. } M=\frac{4mm'}{m+m'}.$$

27. The tension of the rope on each side must be the same, and equal to the force exerted by each man; hence if the man of $11\frac{1}{2}$ stone rise with acceleration f, we have

$$11\tfrac{1}{2}\times 14f=T-11\tfrac{1}{2}\times 14g,$$

and $$12\times 14\times 1=12\times 14g-T.$$

From these equations we have

$$161f+168=168g-161g=224,$$

so that $$f=\frac{224-168}{161}=\frac{8}{23} \text{ ft.-sec. units.}$$

28. Let T be the tension of the rope when the man pulls at it. Since the acceleration of the man, if he did not pull the rope, would be

$$\frac{12-10}{12+10}\cdot g, \text{ i.e. } \frac{g}{11},$$

we have $$168g - T = 168\cdot\frac{g}{22}, \text{ and } T - 140g = 140f,$$

where f is the required acceleration. Hence, by addition, we have

$$f = \frac{8}{55}g.$$

Also the acceleration upwards of the man relative to the rope

$$=\frac{8g}{55}-\frac{g}{22}=\frac{g}{10}.$$

29. The force exerted by the engine may be measured by

$$(16\times 112) \text{ lbs. wt.},$$

and is equal to the resistance overcome; therefore when the mass is detached, the acceleration of the remaining 100 tons becomes f, where

$$16\times 112g = 100\times 2240f + 16\times 100g,$$

so that $$f=\frac{6g}{7000};$$

hence, since 25 miles per hour $=\frac{110}{3}$ feet per second, in 50 seconds the train will have travelled a distance x_1, say,

$$=\frac{110}{3}\times 50 + \frac{1}{2}\cdot\frac{6g}{7000}\cdot(50)^2 = \left[\frac{5500}{3}+\frac{240}{7}\right] \text{ feet.}$$

Also the detached mass has a retardation

$$=\frac{16g}{2240}=\frac{8}{35},$$

and therefore in 50 seconds it travels a distance x_2, say,

$$=\frac{110}{3}\times 50 - \frac{1}{2}\cdot\frac{8}{35}\cdot(50)^2 = \left[\frac{5500}{3}-\frac{2000}{7}\right] \text{ feet;}$$

hence the distance gained on it by the train $= x_1 - x_2 = 320$ feet. Also the detached part comes to rest in

$$\left(\frac{110}{3}\div\frac{8}{35}\right) \text{ seconds, i.e. in } \frac{55\times 35}{12} \text{ seconds,}$$

and then the velocity of the train is

$$\left(\frac{110}{3}+\frac{6g}{7000}\cdot\frac{55\times 35}{12}\right) \text{ feet per second,}$$

i.e. $$\frac{616}{15} \text{ feet per second,}$$

i.e. 28 miles per hour.

30. Let f_1, f_2, f_3 be the accelerations of the masses m, $2m$, and $3m$ along the table and vertically. Let T be the tension of the string.

Then $$mf_1 = T = 2mf_2 \quad\text{.............................}(1),$$

and $$3mg - 2T_1 = 3mf_3 \quad\text{..........................}(2).$$

Also, since the length of the string is constant, the acceleration of the mass $3m$ relative to the mass $2m$ is equal and opposite to the acceleration of the mass $3m$ relative to the mass m.

$$\therefore\ f_3 - f_2 = -(f_3 - f_1),$$

i.e. $$2f_3 = f_1 + f_2 \quad\text{...............................}(3).$$

Now (1) gives $$f_2 = \frac{1}{2} f_1,$$

and (1) and (2) give $$3g - 4f_2 = 3f_3.$$

$$\therefore\ f_2 = \frac{3}{4}(g - f_3);$$

$\therefore$ (3) gives $$2f_3 = 2f_2 + f_2 = 3f_2 = \frac{9}{4}(g - f_3);$$

$$\therefore\ f_3 = \frac{9g}{17}.$$

31. Let f_1 and f_2 be the accelerations of the mass m along the face downwards, and perpendicular to the face; and let f_3 be the acceleration of M. Since the particle and wedge remain in contact, the acceleration of each perpendicular to the face of the wedge is the same.

$$\therefore\ f_2 = f_3 \sin a \quad\text{...............................}(1).$$

Let R be the normal action between the particle and the face of the wedge.

Then $$R - mg\cos a = mf_2 \quad\text{.........................}(2),$$

and $$mg\sin a = mf_1 \quad\text{.........................}(3).$$

Also the horizontal component of R causes the acceleration f_3 in M.

$$\therefore\ -R\sin a = Mf_3 \quad\text{.............................}(4).$$

(1) and (2) give $$R = mf_3 \sin a + mg\cos a,$$

and then (4) gives

$$-mf_3 \sin^2 a - mg\cos a \sin a = Mf_3,$$

i.e. $$f_3 = -\frac{mg\cos a \sin a}{M + m\sin^2 a}.$$

The negative sign shews that the acceleration of the wedge is in an opposite direction to that in which we have assumed it to be. This would be expected *a priori*.

Page 100. Art. 85.

Ex. 1. The velocity is changed from $\frac{88}{8}$ to 44 feet per second. Hence the impulse

$$=9\left(44-\frac{88}{8}\right)=396-264=132 \text{ units.}$$

Ex. 2. We have $P\times\frac{1}{100}=$ the change in the momentum $=2\times 10$. Hence the average value of the force $=2000$ poundals.

Ex. 3. On hitting the floor the velocity of the marble

$$=\sqrt{2g\times 25}=40 \text{ feet per second;}$$

and on leaving it the velocity $=\sqrt{2g\times 16}=32$ feet per second in the opposite direction.

The change is therefore 72 feet per second.

Hence the impulse

$$=\frac{1}{16}\times 72=4\tfrac{1}{2} \text{ units.}$$

Also $$P\times\frac{1}{10}=\frac{1}{16}\times 72,$$

so that the average force causing the destruction of the motion is 45 poundals.

In addition the pressure of the floor supports the weight of the marble, which is about 2 poundals.

EXAMPLES. XIII. (Page 102.)

1. We have $$(7+20)\,V=7\times 10+20\times 2=110,$$

so that $$V=\frac{110}{27}=4\tfrac{2}{27} \text{ feet per second.}$$

2. Here (1) $$(8+24)\,V=8\times 6+24\times 2=96,$$

so that $$V=\frac{96}{32}=3 \text{ feet per second.}$$

(2) $$(8+24)\,V=8\times 6-24\times 2=0,$$

i.e. the compound body is at rest.

3. Here $$(10+12)\,V=10\times 4-12\times 7=-44,$$

so that $$V=-\frac{44}{22}=-2 \text{ feet per second.}$$

4. We have $$\frac{1}{16} \times 1000 = 10V,$$
so that $$V = 6\tfrac{1}{4} \text{ feet per second.}$$

5. We have $$40 \times 2240 \times V = 800 \times 2000,$$
so that $$V = 17\tfrac{6}{7} \text{ feet per second.}$$

6. Let V be the initial velocity of the gun. Since the momentum generated in the shot is equal to that generated in the gun, we have

$$38 \times 2240 \times V = 700 \times 1700,$$

i.e. $$V = \frac{17000}{38 \times 32} \text{ feet per second.}$$

Let $-f$ be the acceleration produced in the gun by the pressure acting on it; then $38f = 17g$, so that

$$f = \frac{17 \times 32}{38}.$$

The required distance s is given by

$$0 = V^2 - 2fs,$$

i.e. $$s = \frac{(17000)^2}{(38 \times 32)^2} \cdot \frac{38}{2 \times 17 \times 32} \text{ feet} = 6{\cdot}8 \text{ feet, nearly.}$$

7. Let V be the initial velocity of the gun. Since the momentum generated in the shot is equal to that generated in the gun, we have

$$81 \times 2240 \times V = 800 \times 1400;$$

so that $$V = \frac{500}{81} \text{ feet per second.}$$

Let $-f$ be the acceleration produced in the gun by the pressure acting on it. Then

$$0 = V^2 - 2f \,.\, 5; \; i.e. \; f = \frac{V^2}{10} = \frac{25000}{(81)^2}.$$

Let P be the magnitude of the pressure; then

$$P = mf = 81 \times 2240 \times \frac{25000}{(81)^2} \text{ poundals} = 9\tfrac{209}{324} \text{ tons wt.}$$

8. The velocity of the gun

$$= \sqrt{2gh} = \sqrt{64 \times 5} = 8\sqrt{5} \text{ feet per second;}$$

so that, if V be the velocity of the shot, we have

$$28V = 2240 \times 7\sqrt{5},$$

whence $$V = 1431 \text{ feet per second, nearly.}$$

EXAMPLES. XIV. (Page 105.)

1. The force to resist the motion

$$=(40\times 50)\text{ lbs. wt.}=2000\text{ lbs. wt}$$

also 30 miles per hour $=$ 44 feet per second; hence if H be the required horse-power, we have

$$H\times 550=2000\times 44,\text{ so that }H=160.$$

2. 40 miles per hour $=\frac{176}{3}$ feet per second. Hence we have

$$H\times 550=2000\times\frac{176}{3},\text{ so that }H=213\tfrac{1}{3}.$$

3. 40 miles per hour $=\frac{176}{3}$ feet per second; and the resistance due to gravity

$$=\frac{1}{200}\times 100\times 2240=1120\text{ lbs. wt.}$$

Hence $$H\times 550=1120\times\frac{176}{3},\text{ so that }H=119\cdot4\dot{6}.$$

4. If the required resistance be x lbs. wt. per ton, then the total resistance

$$=\left(200x+\frac{3}{500}\times 200\times 2240\right)\text{ lbs. wt.}=(200x+2688)\text{ lbs. wt.}$$

Also $$40\text{ miles per hour}=\frac{176}{3}\text{ feet per second.}$$

Hence $$600\times 550=(200x+2688)\times\frac{176}{3},$$

so that $$x=\frac{2937}{200}=14\cdot685\text{ lbs. wt.}$$

5. The total resistance to the motion

$$=\frac{1}{10}\text{ ton wt.}+100\text{ lbs. wt.}=324\text{ lbs. wt.}$$

Also 25 miles per hour $=\frac{110}{3}$ feet per second.

Hence the horse-power H is given by

$$H\times 550=324\times\frac{110}{3},\text{ so that }H=\frac{108}{5}=21\tfrac{3}{5}.$$

6. Since a velocity of $\frac{88}{3}$ feet per second (*i.e.* 20 miles per hour) is acquired in (3×60) seconds, the acceleration f

$$=\frac{88}{3 \times 180}=\frac{22}{135} \text{ ft.-sec. units.}$$

Hence, if P be the force exerted by the engine, in poundals, we have

$$P - 60 \times 10g = 60 \times 2240 \times f,$$

so that $$P = 600\left(32 + 224 \times \frac{22}{135}\right) = \frac{40 \times 9248}{9}.$$

Now the engine must at least be able to exert this force through $\frac{88}{3}$ feet per second. Hence if H be the horse-power, we have

$$H \times 550g = \frac{40 \times 9248}{9} \times \frac{88}{3}, \text{ whence } H = 68\tfrac{68}{135}.$$

7. The height of the plane $=(330 \sin 30^\circ)$ feet $=165$ feet, and the base of the plane $=330 \cos 30^\circ$ feet $=165\sqrt{3}$ feet; hence [cf. Statics, Art. 197] the work expended

$$= 10 \times 2240\left(165 + \frac{1}{\sqrt{3}} \cdot 165\sqrt{3}\right) = 7392000 \text{ ft.-lbs.};$$

$$\text{also the H.-P.} = \frac{22400 \times 330}{30 \times 33000} = 7\cdot 4\dot{6}.$$

8. The distance described during the tenth second

$$= \frac{g}{2}[(10)^2 - 9^2] = 16 \times 19;$$

$$\text{hence the work done} = \frac{1}{2} \times 16 \times 19 = 152 \text{ ft.-lbs.}$$

9. 20 miles per hour $=\frac{88}{3}$ ft. per sec.; hence, if x tons wt be the resistance,

$$(2240x) \times \frac{88}{3} = \text{work done per sec.} = 25000 \times 550,$$

so that $$x = 209\cdot 2.$$

EXAMPLES. XV. (Pages 111, 112.)

1. (1) $\frac{1}{2} \times 10 \times (32)^2$, *i.e.* 5120 units of kinetic energy.

(2) The velocity after half a second

$=32 - \frac{1}{2} \cdot g = 16$, so that the kinetic energy $=\frac{1}{2} \times 10 \times (16)^2 = 1280$.

(3) The velocity after one second $=32 - g = 0$, so that the kinetic energy is zero.

2. The kinetic energy in foot-pounds

$$=\frac{1}{g}\cdot\frac{1}{2}mv^2=\frac{1}{2g}\times 25\times(200)^2=15625.$$

3. The kinetic energy $=\frac{1}{2}\times 10000\times(5000)^2=125\times 10^9$ ergs.

4. The kinetic energy of the ball

$$=\frac{1}{2}\times 5000\times(50000)^2=625\times 10^{10}\text{ ergs.}$$

If V be the velocity of the cannon, then $5000\times 50000 \quad 100000\times V$, so that $V-2500$. Hence the energy of the cannon

$$=\frac{1}{2}\times 100000\times(2500)^2=3125\times 10^8\text{ ergs.}$$

5. The original kinetic energy $=\frac{1}{2}\times\frac{2}{16}\times(1280)^2\times\frac{1}{g}=3200$ ft.-lbs.

Let V be the velocity of the target and bullet, so that

$$\frac{2}{16}\times 1280=\left(10+\frac{2}{16}\right)V;\text{ then } V=\frac{1280}{81}.$$

The final kinetic energy $=\frac{1}{2}\times\frac{162}{16}\times V^2\times\frac{1}{g}=\frac{3200}{81}$ ft.-lbs.

Hence the loss of energy $=3200-\frac{3200}{81}=3160\frac{40}{81}$ ft.-lbs.

6. Ratio of momenta $=\frac{\frac{1}{4}\,.\,1200}{15\,.\,40}=\frac{1}{2}$;

Ratio of energies $=\frac{\frac{1}{2}\,.\,\frac{1}{4}\,.\,1200^2}{\frac{1}{2}\,.\,15\,.\,40^2}=15.$

If P_1, P_2 be the forces in poundals and s_1, s_2 the distances described, then

$$P_1\,.\,1=\text{momentum of shot}=\tfrac{1}{4}\times 1200,$$
$$P_2\,.\,1=\text{momentum of cannon-ball}=15\times 40,$$
$$P_1\;s_1=\text{K.E. of shot}=\tfrac{1}{2}\,.\,\tfrac{1}{4}\,.\,1200^2,$$
and
$$P_2\,.\,s_2=\text{K.E. of cannon-ball}=\tfrac{1}{2}\,.\,15\,.\,40^2$$
$$\therefore\; P_1=300,\; P_2=600,\; s_1=600,\text{ and } s_2=20.$$

EXAMPLES. XVI. (Pages 114—117.)

1. Let v and V be the actual velocities of the shot and the gun respectively. Since the momentum of the one is equal and opposite to that of the other, we have

$$mv = MV \qquad \text{.........(1)}.$$

Also $$u = v + V \qquad \text{.........(2)}.$$

Hence, solving, $v = \dfrac{Mu}{M+m}$, and so $V = \dfrac{mu}{M+m}$.

Also $$\frac{1}{2}mv^2 : \frac{1}{2}MV^2 :: \frac{1}{m} : \frac{1}{M}.$$

2. Let u be the velocity of the shot relative to the gun; this velocity will be in the direction of the bore of the gun. Let V be the horizontal backward velocity of the gun and shot. The total horizontal velocity of the shot is $(u\cos\alpha - V)$, and its vertical velocity is $u\sin\alpha$. Now the impulse of the explosion of the powder on the shot is equal and opposite to its impulse on the gun. Hence the horizontal components of these impulses are the same. Therefore

$$nV = 1\,(u\cos\alpha - V), \text{ and so } V = \frac{u\cos\alpha}{n+1};$$

hence the horizontal velocity of the shot

$$=u\cos\alpha - V=\frac{nu\cos\alpha}{n+1},$$

and therefore

$$\tan\theta=\frac{\text{vertical velocity of the shot}}{\text{horizontal velocity of the shot}}=\frac{u\sin\alpha\,(n+1)}{nu\cos\alpha}$$

$$=\tan\alpha\left(1+\frac{1}{n}\right).$$

3. If P be the resistance, we have

$$P\times\frac{1}{100}=\text{momentum destroyed}=\frac{1}{2}\times 800\times 2240$$

$$=896000 \text{ units of impulse.}$$

The kinetic energy

$$=\frac{1}{2}\times 1120\times(800)^2 \text{ ft.-poundals}$$

Therefore, if s feet be the distance in which this is destroyed by P, we have

$$Ps=\frac{1}{2}\times 1120\times(800)^2,$$

i.e.

$$s=\frac{560\times(800)^2}{896000\times 100}=4 \text{ feet.}$$

4. The velocity of the mass on hitting pile $=\sqrt{2g\ 10}=\sqrt{640}$ If v be the velocity of the pile and mass just after the blow, and x be the distance required and t the time, then

$$(4+12)\,v=4\sqrt{640}, \text{ whence } v=\sqrt{40}.$$

$$\left(\frac{3}{2}-\frac{16}{20}\right)gs=\frac{1}{2}\cdot\frac{16}{20}\cdot v^2, \text{ whence } s=\frac{5}{7}\text{ ft.};$$

and

$$\left(\frac{3}{2}-\frac{16}{20}\right)gt=\frac{16}{20}v, \text{ whence } t=\frac{1}{14}\sqrt{10}\text{ sec.}$$

$$\text{Also required fraction}=\frac{\frac{1}{2}\cdot\frac{4}{20}\cdot 640-\frac{1}{2}\cdot\frac{16}{20}v^2}{\frac{1}{2}\cdot\frac{4}{20}\cdot 640}=\frac{3}{4}.$$

5. If V be the required velocity, we have

$$20\times 20000=V(20+50000),$$

so that $V=7{\cdot}997$ centimetres per second, nearly.

Also if f be the average acceleration of the bullet in the barrel,

$$(20000)^2=2f\times 100;$$

and if P be the required force in dynes, $P=mf=2000000\times 20$

$$=\frac{40000000}{981}\text{ grammes wt.}=40775\text{ grammes wt., nearly.}$$

6. The velocity of the hammer on striking the iron

$$=\sqrt{2g\,.\,4}=16 \text{ feet per second},$$

so that the momentum $=4\times 112\times 16=Pt.$

Hence $P=(4\times 112\times 16\times 50)$ poundals + wt. of hammer

$$=\left(\frac{4\times 112\times 16\times 50}{32}+4\times 112\right)\text{lbs. wt.}=104 \text{ cwt.}$$

7. The acceleration at first

$$=\frac{2m-m}{2m+m}g=\frac{g}{3};$$

therefore the common velocity at the end of 3 seconds $=g$. Also on m picking up a mass m, the acceleration becomes zero; and the common velocity V and the impulse T' on the string are given by the changes of momentum on each side; hence

$$T'=-mg+2mV, \text{ and } T'=2m(g-V),$$

whence $V=24$ feet per second.

8. Until A begins to move the acceleration

$$=\frac{m}{m+m}g \text{ (Art. 75)}=\frac{g}{2}.$$

Hence, if v_1 be the velocity just before A moves, then

$$v_1^2=2\,.\,\frac{g}{2}\,.\,3=3g.$$

If v_2 be the velocity just after A moves, we have, since the total momentum is not altered by the jerk,

$$3v_2=2v_1.$$

$$\therefore\ v_2=\frac{2}{3}\sqrt{3g}=\frac{2}{3}\sqrt{96}=\frac{8}{3}\sqrt{6} \text{ feet per second.}$$

When A is in motion, the common acceleration is $\frac{g}{3}$. Hence, if v_3 be the velocity when B reaches the edge of the table, we have

$$v_3^2=v_2^2+2\,.\,\frac{g}{3}\,.\,\frac{1}{2};$$

$$\therefore\ v_3^2=\frac{64}{9}\times 6+\frac{32}{3},$$

so that $v_3=\frac{4}{3}\sqrt{30}$ feet per second.

9. At first the acceleration $=\frac{7-5}{7+5}\cdot g=\frac{16}{3}$; therefore at the end of 3 seconds the velocity of the mass of 5 lbs. is 16 feet per second upwards, and the velocity of the mass of 7 lbs. is zero; hence if the mass of 5 lbs. rise and return to its position in t seconds more, we have

$$0=16t-\frac{1}{2}gt^2,$$

so that $t=1$ second. The string then becomes tight with a jerk, no momentum being lost, and the system starts (the mass of 7 lbs. upwards and the mass of 5 lbs. downwards) with velocity u given by

$$(5+7)\,u=5\times 16,\ \ i.e.\ \ u=\frac{20}{3}\text{ feet per second,}$$

and the retardation is $\frac{16}{3}$ as above; hence

$$0=u-ft\text{ gives }t=\frac{20}{3}\div\frac{16}{3}=1\tfrac{1}{4}\text{ second,}$$

when the system is for an instant at rest; thus the required time

$$=1+1\tfrac{1}{4}=2\tfrac{1}{4}\text{ seconds.}$$

10. Before the string becomes taut the acceleration

$$=\frac{5-3}{5+3}\cdot g=\frac{g}{4}.$$

Hence the required time t is given by

$$1=\frac{1}{2}\cdot\frac{g}{4}\cdot t^2,\text{ so that }t=\sqrt{\frac{8}{32}}=\frac{1}{2}.$$

The common velocity just before the string becomes taut

$$=\frac{1}{2}\cdot\frac{g}{4}=4\text{ feet per second.}$$

Hence the common velocity just after the string becomes taut

$$=\frac{8}{10}\times 4=3\cdot 2\text{ feet per second.}$$

After the string is taut, there is no acceleration, and the system will move uniformly with the velocity of 3·2 feet per second

11. Let u be the common velocity at the instant when P is stopped. The mass Q continues to move with velocity u and acceleration $-g$; also the mass P falls freely. The string becomes taut when both masses have described the same distance, *i.e.* after a time t given by

$$ut-\frac{1}{2}gt^2=\frac{1}{2}gt^2,\ \ i.e.\text{ when }\ t=\frac{u}{g}.$$

12. Let u be the velocity of the mass M just before the string becomes tight, so that $u^2=2ga$. Immediately after the string becomes tight, let v be the common velocity of the two masses. The impulsive tension is equal to $M(u-v)$, and also to mv. Hence

$$M(u-v)=mv, \text{ so that } v=\frac{Mu}{M+m}.$$

The acceleration f of m is $\frac{m-M}{m+M}\cdot g$.

Hence the required time is twice the time in which v will be destroyed by f, *i.e.* it $=\frac{2v}{f}=\frac{2Mu}{(m-M)g}=\frac{2M}{m-M}\sqrt{\frac{2a}{g}}$.

Also required fraction

$$=\frac{\frac{1}{2}Mu^2-\frac{1}{2}(M+m)v^2}{\frac{1}{2}Mu^2}=1-\frac{M+m}{M}\cdot\frac{v^2}{u^2}=1-\frac{M}{M+m}=\frac{m}{M+m}.$$

13. When the bar is on, the acceleration of the 9 oz. mass is

$$\frac{16-12}{16+12}g, \text{ i.e. } \frac{g}{7}.$$

The velocity acquired in descending 7 feet

$$=\sqrt{2\cdot\frac{g}{7}\cdot 7}=\sqrt{2g}=8 \text{ feet per second.}$$

After the removal of the bar, the retardation is

$$\frac{12-9}{12+9}g, \text{ i.e. } \frac{g}{7},$$

as before; thus the velocity will be destroyed in the next 7 feet. Also

$$s=\frac{1}{2}ft^2 \text{ gives } 7=\frac{1}{2}\cdot\frac{g}{7}\cdot t^2,$$

so that $$t=\frac{7}{4}\text{ sec.}$$

Therefore the mass of 9 oz. returns to the ring in $3\times\frac{7}{4}$, *i.e.* $5\frac{1}{4}$ seconds after start. Then a jerk takes place, and if v be the resulting velocity of the system, we have

$$(12+7+9)v=(12+9)8,$$

so that $v=6$ feet per second. Also

$$v=ft \text{ gives } t=6\div\frac{g}{7}=\frac{21}{16}\text{ sec.}$$

Therefore the mass of 9 oz. with the bar ascends through the ring a second time after

$$4\times\frac{21}{16}, \text{ i.e. } \frac{21}{4}, \text{ i.e. } 5\frac{1}{4} \text{ seconds.}$$

On the second jerk, the velocity is diminished to

$$\frac{3}{4}\times 6, \text{ i.e. } 4\tfrac{1}{2} \text{ feet per second;}$$

and $$v=ft \text{ gives } t=\frac{3}{4}\times\frac{21}{4} \text{ seconds.}$$

Hence the whole time

$$=5\tfrac{1}{4}+\left(\frac{21}{4}+\frac{3}{4}\cdot\frac{21}{4}+\frac{3^2}{4^2}\cdot\frac{21}{4}+\ \ldots\text{ad inf.}\right),$$

$$=\frac{21}{4}+\frac{21}{4}\cdot\frac{1}{1-\frac{3}{4}}=\frac{21}{4}+\frac{21}{4}\cdot 4=26\tfrac{1}{4} \text{ seconds.}$$

14. If the masses of the carriages and of the two loads of passengers be the same and they have no acceleration, then the passengers in each carriage have the same velocity as that carriage; hence an interchange of passengers increases the momentum of the slower carriage, and diminishes the momentum of the faster carriage. Thus the effect of each interchange is to increase the velocity of the slower and diminish the velocity of the faster carriage. If this continue, either the velocities become equal at some interchange or the slower becomes the faster carriage, and then the process is reversed, so that the carriages tend to have the same velocity.

15. The man raises 210 lbs. through 11000 feet in 7 hours. Hence the amount of work he does per minute

$$=\frac{210\times 11000}{7\times 60}=5500 \text{ ft. lbs.}$$

Also the horse raises 168 lbs. through 11000 feet in 56 minutes. Hence the amount of work it does per minute

$$=\frac{168\times 11000}{56}=33000 \text{ ft.-lbs}$$

Hence in the same time the horse does six times as much work as the man.

16. The velocity of the sledge would be acquired by falling through 16 feet.

Hence the work done per minute

$$=(25\times 14\times 16) \text{ ft.-lbs.}$$

Hence the rate of working

$$=\frac{25\times 14\times 16}{33000}=\cdot 1\dot{6}\dot{9} \text{ H.P.}$$

17. The total force on the hammer is equal to the weight of 50 tons, so that its acceleration is

$$g+\frac{3}{2}g, \quad i.e. \quad \frac{5}{2}g.$$

Hence, if v be the required velocity, we have

$$v^2=2\,.\,\frac{5g}{2}\,.\,5, \text{ so that } v=20\sqrt{2} \text{ feet per second.}$$

Also the work done on it

$$=(5\times 50) \text{ ft.-tons}=560000 \text{ ft.-lbs.}$$

18. 50 miles per hour $=\left(\frac{5}{3}\times 44\right)$ feet per second.

Hence, if f be the retardation,

$$\left(\frac{5}{3}\times 44\right)^2=2f\times 363\times 3, \text{ so that } f=\frac{200}{81};$$

hence the resistance $=\left(\frac{200}{81}\times\frac{150}{g}\right)$ tons wt. $=11\frac{31}{54}$ tons wt.

The work done

$$=(11\tfrac{31}{54}\times 2240\times 363\times 3) \text{ ft.-lbs.}=28233333\tfrac{1}{3} \text{ ft.-lbs.}$$

19. The acceleration f with which the train must move so that a velocity of 44 feet per second is destroyed in a quarter of a mile is given by

$$44^2=2f\times 1320, \quad i.e. \quad f=\frac{11}{15}.$$

Let x tons be the weight of the brake van.
Then the total force on the train in poundals

$$=200\times 2240\times\frac{g}{100}+x\times 2240g\times\frac{1}{6}+(200-x)\,8g.$$

But this force must equal

$$200\times 2240\times f, \quad i.e. \quad 200\times 2240\times\frac{11}{15} \text{ poundals.}$$

Equating these two quantities and solving, we have $x=11\frac{63}{137}$ tons.

20. 12 miles per hour $=\frac{88}{5}$ feet per second.

Let the resistance be x lbs. wt.

Then $x\times\frac{88}{5}=\frac{1}{10}\times 550$, so that $x=3{\cdot}125$ lbs. wt.

Up the incline the total resistance

$$=\left(3{\cdot}125+\frac{168}{50}\right)=6\tfrac{97}{200} \text{ lbs. wt.}$$

Hence, if v be the velocity up the incline, we have

$$6\tfrac{97}{200} \times v = \frac{1}{10} \times 550,$$

so that
$$v = \frac{11000}{1297} \text{ feet per second};$$

and the velocity of the man

$$= \left(\frac{11000}{1297} \times \frac{60 \times 60}{1760 \times 3}\right) \text{ miles per hour} = \text{about } 5{\cdot}8 \text{ miles per hour.}$$

21. At each stroke the driving-wheel turns half round, and it makes $\dfrac{12 \times 1760 \times 3}{12}$ revolutions per hour, *i.e.* $\dfrac{1760 \times 3}{60}$ per minute, *i.e.* 88 per minute; therefore the man makes 176 strokes per minute, and therefore the work he does $= 20 \times 1 \times 176 = 3520$ ft. lbs. per minute.

22. If m be the mass of the bullet and u be its initial velocity, then after passing through one plank the velocity $= \dfrac{19}{20}u$; therefore the kinetic energy lost

$$= \frac{1}{2}m\left[u^2 - \left(\frac{19}{20}u\right)^2\right] = \frac{1}{2}m \cdot \frac{39}{400} \cdot u^2;$$

and an equal quantity of kinetic energy being lost on passing through each plank, the number x of them required is given by

$$\frac{1}{2}mu^2 = \frac{1}{2}m \cdot \frac{39}{100} \cdot u^2 \cdot x, \text{ whence } x = 10\tfrac{10}{39}.$$

23. The velocity $= \dfrac{n \times 1760 \times 3}{60} = 88n$ feet per minute; hence the work done by the resistance $= R \times 88n$ ft.-lbs. per minute; hence, if the man take x strokes per minute, $Ex = 18nR$, whence

$$x = \frac{88nR}{E}.$$

24. If R be the resistance and P the thrust on the pedal, we have

$$\pi \cdot \frac{70}{12} \cdot R = \text{work done in one revolution of the pedals}$$

$$= 2 \cdot \frac{13\frac{1}{2}}{12} \cdot P = \frac{1}{10} \times \frac{1}{60}\text{ H.P.} = \frac{33000}{600} \text{ ft.-lbs.}$$

$$\therefore\ R = 3 \text{ lbs. wt. and } P = 24\tfrac{4}{9} \text{ lbs. wt.}$$

25. If P be the frictional resistance, then

$$P=\frac{1}{60}\times 180 \text{ lbs. wt.}=3 \text{ lbs. wt.},$$

since there is no acceleration on the incline.

Also in going up the incline at 8 miles per hour, *i.e.* $\frac{176}{15}$ ft. per sec., the work done per sec.

$$=\frac{176}{15}\left[P+\frac{180}{100}\right] \text{ ft.-lbs.}=\frac{176}{15}\times\frac{480}{100} \text{ ft.-lbs.}$$

$$\therefore \text{ H.P.}=\frac{176}{15}\times\frac{480}{100}\times\frac{1}{550}=\cdot 1024.$$

26. In each second the momentum destroyed

$$=\frac{200}{60}\times 10=33\tfrac{1}{3} \text{ units.}$$

$$\therefore \text{ Force}=33\tfrac{1}{3} \text{ poundals}=\frac{33\frac{1}{3}}{32} \text{ lbs. wt.}=1\tfrac{1}{24} \text{ lbs. wt.}$$

Energy delivered per second

$$=\frac{1}{2}\cdot\frac{200}{60}\cdot 10^2 \text{ units of energy}=\frac{1000}{6\times 32} \text{ ft.-lbs.}$$

$$\therefore \text{ Rate}=\frac{1000}{6\times 32\times 550} \text{ H.P.}=\frac{5}{528} \text{ H.P.}$$

27. If V be the velocity of the hammer and nail just after the impact, the principle of Conservation of Momentum gives

$$3\times 8=\left(3+\frac{1}{8}\right)V, \text{ i.e. } V=\frac{192}{25}.$$

If R be the resistance in poundals, then

$$R\times\frac{1}{24}=\text{work done by resistance}=\frac{1}{2}\times\frac{25}{8}\times\left(\frac{192}{25}\right)^2.$$

$$\therefore R=\frac{3}{2}\cdot\frac{192^2}{25} \text{ poundals}=\frac{3}{2}\times\frac{192\times 6}{25} \text{ lbs. wt.}=69\cdot 12 \text{ lbs. wt.}$$

Page 119. Art. 99.

Ex. 2. The acceleration of $3m$ downwards and of m upwards

$$=\frac{3m-m}{3m+m}g=\frac{g}{2};$$

hence the acceleration of their centre of inertia *downwards*

$$=\frac{3m.\frac{g}{2}-m.\frac{g}{2}}{3m+m}=\frac{g}{4}.$$

Ex. 3. (1) The velocity is in the direction in which both masses are moving and

$$=\frac{6\times3+4\times8}{6+4}=5 \text{ feet per second.}$$

(2) The velocity

$$=\frac{6\times8\sim4\times8}{6+4}=\frac{32-18}{10}=1\tfrac{2}{5} \text{ feet per second}$$

in the direction in which the second body is moving.

Ex. 4. If the masses start from the point O, and P and Q be the positions of mn and m respectively after any time t, then

$$OP=vt \text{ and } OQ=nvt;$$

also, if G be their centre of inertia,

$$PG:GQ=m:mn=1:n=OP:OQ,$$

i.e. G always lies on the straight line bisecting the angle between the two given straight lines.

Ex. 5. Since the velocity in any direction of the centre of inertia of any system is equal to the total momentum in that direction divided by the total mass, it follows that in one case the velocity of the centre of inertia along each of the given straight lines is constant. Compounding therefore these two component velocities we have a constant velocity in a constant direction.

EXAMPLES. XVII. (Pages 127, 128.)

1. (1) $h=\frac{u^2\sin^2\alpha}{2g}=\frac{(64)^2}{2\times32}\cdot\left(\frac{1}{2}\right)^2=16$ feet;

$$t=\frac{2u\sin\alpha}{g}=\frac{2\times64}{32}\cdot\left(\frac{1}{2}\right)=2 \text{ seconds};$$

$$R=\frac{u^2\sin2\alpha}{g}=\frac{(64)^2}{32}\cdot\left(\frac{\sqrt{3}}{2}\right)=110\cdot8 \text{ feet.}$$

(2) $h=\frac{(80)^2}{2\times 32}\cdot\left(\frac{\sqrt{3}}{2}\right)^2=75$ feet;

$t=\frac{2\times 80}{32}\cdot\left(\frac{\sqrt{3}}{2}\right)=4\cdot 33$ seconds;

$R=\frac{(80)^2}{32}\cdot\left(\frac{\sqrt{3}}{2}\right)=173\cdot 2$ feet.

(3) $h=\frac{(96)^2}{2\times 32}\cdot\cos^2 15^\circ=144\times\frac{1+\cos 30^\circ}{2}=72\left(1+\frac{\sqrt{3}}{2}\right)$

$=134\cdot 34$ feet, nearly;

$t=\frac{2\times 96}{32}\times\cdot 96592=5\cdot 795$ seconds;

$R=\frac{(96)^2}{32}\cdot\left(\frac{1}{2}\right)=144$ feet.

(4) $h=\frac{(200)^2}{2\times 32}\cdot\left(\frac{3}{5}\right)^2=225$ feet;

$t=\frac{2\times 200}{32}\cdot\left(\frac{3}{5}\right)=7\frac{1}{2}$ seconds;

$R=\frac{(200)^2}{32}\times 2\times\frac{3}{5}\times\frac{4}{5}=1200$ feet.

2. (1) $R=\frac{u^2}{g}=\frac{(48)^2}{32}=72$ feet.

(2) $R=\frac{(60)^2}{32}=112\frac{1}{2}$ feet.

(3) $R=\frac{(100)^2}{32}=312\frac{1}{2}$ feet.

3. The angle of projection is 45°, since the horizontal range is a maximum.

$$\therefore\ \text{distance}=\frac{V^2}{g}\ (\text{Art. }106)=\frac{(16000)^2}{981}\ \text{cms.}$$

$$=\frac{2560000}{981}\ \text{metres}=2609\cdot 58\ \ldots\ \text{metres.}$$

Also greatest height $=\frac{V^2\sin^2 45^\circ}{2g}=\frac{1}{4}\cdot\frac{V^2}{2g}=652\cdot 39\ \ldots$ metres.

4. $8000=\frac{V^2}{g}$, so that $V=\sqrt{8000\times 981}$ cms. per sec.

Time $=\frac{2\,.\,V\sin 45^\circ}{g}=\frac{\sqrt{2}\,.\,V}{981}=\sqrt{\frac{16000}{981}}$ secs. $=4\cdot 04$ secs.

Height $=\frac{V^2\sin^2 45^\circ}{2g}=\frac{1}{4}\cdot\frac{V^2}{g}=2000$ cms. $=20$ metres.

5. If $$\tan^{-1} 3 = \alpha,$$
then $$\sin\alpha = \frac{3}{\sqrt{10}} \text{ and } \cos\alpha = \frac{1}{\sqrt{10}}.$$
The required height
$$= \frac{(80)^2}{2 \times 32} \cdot \left(\frac{9}{10}\right) = 90 \text{ feet.}$$
Also, if θ be the inclination to the horizon at a height of 60 feet, we have, by Art. 103,
$$\tan\theta = \sqrt{(80)^2 \cdot \frac{9}{10} - 2 \cdot 32 \cdot 60} \div 80 \cdot \frac{1}{\sqrt{10}} = \sqrt{3}, \textit{ i.e. } \theta = 60^\circ.$$
Again, the required time
$$= \frac{2 \times 80}{32} \cdot \frac{3}{\sqrt{10}} = \frac{15\sqrt{10}}{10} = 4\tfrac{3}{4} \text{ seconds, nearly.}$$

6. If t be the time of flight, then
$$9 = \frac{1}{2} g t^2, \text{ so that } t = \frac{3}{4} \text{ second.}$$
Hence $1000 = ut$, so that
$$u = 1000 \times \frac{4}{3} = 1333\tfrac{1}{3} \text{ feet per second.}$$

7. The vertical velocity of the stone on reaching the ground being due to the height h is $\sqrt{2gh}$; hence if t be the time of flight,
$$gt = \sqrt{2gh}, \text{ so that } t = \sqrt{\frac{2h}{g}},$$
$$\text{and the range} = \sqrt{2gh} \times \sqrt{\frac{2h}{g}} = 2h,$$
the horizontal velocity being constant; also the required velocity
$$= \sqrt{2gh + 2gh} = 2\sqrt{gh}.$$

8. The vertical velocity of the stone due to falling through the height 9 feet
$$= \sqrt{2g \cdot 9} = 24 \text{ feet per second;}$$
and the horizontal velocity of the stone is the same as that of the carriage, *i.e.* 44 feet per second; hence the resultant velocity of the stone
$$= \sqrt{(24)^2 + (44)^2} = 50{\cdot}1 \text{ feet per second;}$$
and the direction of the stone's motion is $\tan^{-1}\frac{24}{44}$, *i.e.* $\tan^{-1}\frac{6}{11}$, *i.e.* 28° 36′ to the horizon.

9. (1) The vertical velocity of the body

$$=gt=64 \text{ feet per second};$$

hence the required velocity

$$=\sqrt{(64)^2+(16)^2}=16\sqrt{17}\doteq 65{\cdot}97 \text{ feet per second},$$

at an angle $\tan^{-1}\frac{64}{16}$, *i.e.* $\tan^{-1}4$, *i.e.* $75^\circ\ 58'$ with the horizon.

(2) The vertical velocity

$$=\sqrt{2g\,.\,144}=96 \text{ feet per second};$$

hence the required velocity

$$=\sqrt{(96)^2+(16)^2}=16\sqrt{37}=97{\cdot}31 \text{ feet per second},$$

at an angle $\tan^{-1}\frac{96}{16}$, *i.e.* $\tan^{-1}6$, *i.e.* $80^\circ\ 32'$ with the horizon.

10. The initial vertical velocity

$$=768\sin 30^\circ=384 \text{ feet per second}.$$

Hence the time t that elapses before the shot hits the water is given by

$$-400=384t-\frac{1}{2}gt^2,$$

i.e. by $\quad t^2-24t-25=0,\quad$ *i.e.* $t=25$ seconds.

Hence the required distance

$$=(25\times 768\cos 30^\circ) \text{ feet}=25\times 384\sqrt{3} \text{ feet}=5543 \text{ yards, nearly}.$$

11. If t seconds be the time of flight, and x feet be the required distance, then

$$-\frac{13}{2}g=6g\,.\,t-\frac{1}{2}gt^2,$$

so that $t=13$ seconds. Also $x=8g\,.\,13=3328$ feet.

12. Let u be the velocity of projection. Since the shot is moving horizontally at the height of 14100 cms., that is its greatest height; and if

$$\alpha=\cot^{-1}5, \text{ then } \sin\alpha=\frac{1}{\sqrt{26}}.$$

Hence we have

$$\frac{u^2\sin^2\alpha}{2g}=14100, \quad i.e. \quad u^2=2\times 981\times 26\times 14100,$$

so that $\qquad u=26800$ cms. per second, nearly.

13. The horizontal range is 300 feet, and the greatest height is 75 feet, so that

$$\frac{2u^2 \sin\alpha \cos\alpha}{g} = 300, \text{ and } \frac{u^2 \sin^2\alpha}{2g} = 75.$$

Hence, by division, $\tan\alpha = 1$, *i.e.* $\alpha = 45°$, and $u^2 = 4 \times 32 \times 75 = 9600$,

i.e. $u = 40\sqrt{6}$ feet per second $= 97{\cdot}98$ feet per second, nearly.

14. If u be the velocity of projection, we have

$$4 \times 1760 \times 3 = \frac{2u^2}{32} \cdot \frac{4}{5} \cdot \frac{3}{5},$$

whence $u = 80\sqrt{110} = 839{\cdot}04$ feet per second;

and the velocity at the highest point $=$ the horizontal velocity

$$= u \cos\alpha = u \times \frac{3}{5} = 48\sqrt{110} = 503{\cdot}4 \text{ feet per second.}$$

15. If u_1 and u_2 be the velocities of projection and h be the greatest height attained by each ball, then

$$u_1^2 \sin^2 60° = 2gh, \text{ and } u_2^2 \sin^2 30° = 2gh,$$

so that $3u_1^2 = u_2^2$, *i.e.* $u_1 : u_2 = 1 : \sqrt{3}$.

If they have the same horizontal range R, then

$$\frac{2}{g} u_1 \sin 60° \cdot u_1 \cos 60° = R = \frac{2}{g} \cdot u_2 \sin 30° \cdot u_2 \cos 30°,$$

and so $u_1 = u_2$.

16. If u be the velocity of projection at an elevation α, and h be the greatest height, then $u^2 \sin^2\alpha = 2gh$,

and the velocity at height h is $u\cos\alpha$; also the velocity at height $\frac{h}{2}$

$$= \sqrt{u^2 - 2g \cdot \frac{h}{2}} = \sqrt{u^2 - gh}.$$

$$u \cos\alpha = \sqrt{\frac{2}{5}} \cdot \sqrt{u^2 - gh} = \sqrt{\frac{2}{5}} \cdot \sqrt{u^2 - \frac{u^2 \sin^2\alpha}{2}}.$$

$$\therefore \cos^2\alpha = \frac{2}{5}\left(1 - \frac{\sin^2\alpha}{2}\right) = \frac{1}{5}(1 + \cos^2\alpha).$$

$$\therefore 4\cos^2\alpha = 1, \text{ i.e. } \alpha = 60°.$$

17. If α be the angle of projection, then

(1) $\dfrac{u^2 \sin 2\alpha}{g} = \dfrac{4u^2 \sin^2\alpha}{2g}$, *i.e.* $\cos\alpha = \sin\alpha$.

$$\therefore \cot\alpha = 1, \text{ i.e. } \alpha = 45°.$$

(2) $\dfrac{u^2 \sin 2\alpha}{g} = \dfrac{4\sqrt{3}u^2 \sin^2\alpha}{2g}$.

$$\therefore \cot\alpha = \sqrt{3}, \text{ i.e. } \alpha = 30°.$$

18. Here $$\frac{u^2 \sin 2a}{g} = \frac{u^2}{2g}.$$

$$\therefore \sin 2a = \frac{1}{2} = \sin 30^\circ, \text{ or } \sin 150^\circ,$$

i.e. $$a = 15^\circ, \text{ or } 75^\circ.$$

EXAMPLES. XVIII. (Page 133.)

1. $$\text{The range} = 2 \cdot \frac{(600)^2}{g} \cdot \frac{\cos 60^\circ \sin 30^\circ}{\cos^2 30^\circ} \text{ feet} = 2500 \text{ yards.}$$

$$\text{The time of flight} = 2 \cdot \frac{600}{g} \cdot \frac{\sin 30^\circ}{\cos 30^\circ} = 21 \cdot 7 \text{ seconds.}$$

2. $$\text{The range} = \frac{2V^2}{g} \cdot \frac{\cos 75^\circ \sin 45^\circ}{\cos^2 30^\circ} = \frac{V^2}{48}(\sqrt{3} - 1).$$

$$\text{The maximum range} = \frac{V^2}{g} \cdot \frac{1}{1 + \sin 30^\circ} = \frac{V^2}{48}.$$

3. $$\text{The time of flight} = \frac{2 \times 64}{g} \cdot \frac{\sin 15^\circ}{\cos 30^\circ} = 4 \times \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{2}{\sqrt{3}}$$

$$= \frac{2}{3}(3\sqrt{2} - \sqrt{6}) = 1 \cdot 2 \text{ second, nearly.}$$

$$\text{The range} = \frac{2 \times (64)^2}{g} \cdot \frac{\cos 45^\circ \sin 15^\circ}{\cos^2 30^\circ} = 62 \cdot 5 \text{ feet, nearly.}$$

$$\text{The greatest range} = \frac{(64)^2}{g} \cdot \frac{1}{1 + \sin 30^\circ} = 85\tfrac{1}{3} \text{ feet.}$$

4. Let β be the inclination of the plane, so that

$$\sin \beta = \frac{3}{5} \text{ and } \cos \beta = \frac{4}{5}.$$

(i) $$\text{The range} = 2 \cdot \frac{(1280)^2}{g} \cdot \frac{\cos 45^\circ \sin (45^\circ - \beta)}{\cos^2 \beta}$$

$$= \frac{(1280)^2}{16} \cdot \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} \cdot \frac{4}{5} - \frac{1}{\sqrt{2}} \cdot \frac{3}{5}\right)}{\left(\frac{16}{25}\right)} = 16000 \text{ feet.}$$

(ii) $$\text{The range} = 2 \cdot \frac{(1280)^2}{g} \cdot \frac{\cos 45^\circ \sin (45^\circ + \beta)}{\cos^2 \beta} = 112000 \text{ feet.}$$

5. The greatest range $=\dfrac{u^2}{g}\cdot\dfrac{1}{1+\sin\beta}$; and the corresponding time of flight

$$=\frac{2u}{g}\cdot\frac{\sin\left(45^\circ+\frac{\beta}{2}-\beta\right)}{\cos\beta}=\frac{2u}{g}\cdot\frac{\sin\left(45^\circ-\frac{\beta}{2}\right)}{\cos\beta}.$$

Hence (1) the greatest range

$$=\left[\frac{(800)^2}{32}\cdot\frac{1}{1+\sin 45^\circ}\right]\text{ feet}=20000\,(2-\sqrt{2})=11716\text{ feet, nearly.}$$

The time of flight

$$=\left[\frac{2\times 800}{32}\cdot\frac{\sin(45^\circ-22\tfrac{1}{2}^\circ)}{\cos 45^\circ}\right]\text{seconds}=50\sqrt{2}\times\sin 22\tfrac{1}{2}^\circ$$

$$=50\sqrt{2}\times\cdot 3826834=27\text{ seconds, nearly.}$$

(2) The greatest range $=\left[\dfrac{(800)^2}{32}\cdot\dfrac{1}{1+\sin 60^\circ}\right]$ feet

$$=40000\,(2-\sqrt{3})=40000\,(2-1\cdot 73205)=10718\text{ feet, nearly.}$$

The time of flight $=\left[\dfrac{2\times 800}{32}\cdot\dfrac{\sin(45^\circ-30^\circ)}{\cos 60^\circ}\right]$ seconds

$$=50\times 2\times\sin 15^\circ-100\times\cdot 2588190=25\cdot 9\text{ seconds, nearly.}$$

(3) The greatest range

$$=\frac{800^2}{32}\,\frac{1}{1+\frac{1}{20}}=20000\times\frac{20}{21}=19048\text{ feet, nearly.}$$

Also $$\cos\beta=\sqrt{1-\frac{1}{400}}=1-\frac{1}{800}\text{ nearly.}$$

$$\therefore\ \sin\frac{\beta}{2}=\sqrt{\frac{1}{2}(1-\cos\beta)}=\sqrt{\frac{1}{1600}}\text{ nearly}=\frac{1}{40}\text{ nearly,}$$

and $$\cos\frac{\beta}{2}=\sqrt{\frac{1}{2}(1+\cos\beta)}=\sqrt{1-\frac{1}{1600}}=1-\frac{1}{3200}\text{ nearly.}$$

Hence the time

$$=\frac{2\times 800}{g}\,\frac{\frac{1}{\sqrt{2}}\left[1-\frac{1}{3200}-\frac{1}{40}\right]}{1-\frac{1}{800}}$$

$$=\frac{50}{\sqrt{2}}\left(1-\frac{81}{3200}\right)\left(1+\frac{1}{800}\right)\text{ nearly,}$$

$$=25\,.\,\sqrt{2}\,.\left(1-\frac{77}{3200}\right)=34\cdot 4\text{ seconds, nearly.}$$

(4) The greatest range

$$=\frac{800^2}{32}\frac{1}{1+\frac{5}{13}}=20000\times\frac{13}{18}=14444\tfrac{4}{9}\text{ feet.}$$

Also $$\cos\beta=\sqrt{1-\frac{25}{169}}=\frac{12}{13}.$$

$$\therefore\ \sin\frac{\beta}{2}=\sqrt{\frac{1}{2}(1-\cos\beta)}=\frac{1}{\sqrt{26}},\text{ and }\cos\frac{\beta}{2}=\frac{5}{\sqrt{26}}.$$

Hence the time

$$=\frac{2\times800}{g}\frac{\frac{1}{\sqrt{2}}\left[\frac{5}{\sqrt{26}}-\frac{1}{\sqrt{26}}\right]}{\frac{12}{13}}=\frac{100}{12}\sqrt{13}=30\text{ seconds, nearly.}$$

6. By Art. 106 we have

$$\frac{u^2}{g}=15000\text{ feet.}$$

Hence the greatest range up the plane

$$=15000\times\frac{1}{1+\sin45^\circ}\text{ feet}$$
$$=15000\,(2-\sqrt{2})\text{ feet}=2929\text{ yards, nearly.}$$

Also the greatest range down the plane

$$=\frac{15000}{1-\sin45^\circ}=15000\,(2+\sqrt{2})\text{ feet}=17071\text{ yards, nearly}$$

7. Here $$\frac{u^2}{g}=1000;$$

hence the greatest ranges are

$$\frac{1000}{1\pm\sin30^\circ}\text{ metres,}$$

i.e. $666\frac{2}{3}$ and 2000 metres respectively.

8. (1) The range

$$=\frac{2u^2}{g}\frac{\cos(90^\circ+30^\circ)\sin90^\circ}{\cos^2 30^\circ}\text{ metres}$$
$$=-84{\cdot}95\text{ metres, } i.e.\ 84{\cdot}95\text{ metres down the plane.}$$

(2) The range $$=-\frac{2\times25^2}{9{\cdot}81}\cdot\frac{\sqrt{3}}{2}\cdot\frac{4}{1}\text{ metres,}$$

i.e. 441·4 metres down the plane.

EXAMPLES. XIX. (Pages 138—141.)

1. The space of country is a circle of which the radius is the greatest range of the guns on a horizontal plane, *i.e.* $\dfrac{(\text{velocity})^2}{g'}$, where g' is the acceleration of the moon's attraction; hence this radius

$$=\left[(1600)^2 \div \frac{g}{6}\right] \text{ feet} = 90\tfrac{10}{11} \text{ miles.}$$

2. Let u be the velocity and α be the angle of projection; then $u \cos \alpha$ is the horizontal velocity; and, if t seconds be the time of reaching the net, we have

$$39 = u \cos \alpha \,.\, t, \text{ so that } t = \frac{39}{u \cos \alpha};$$

also

$$3\tfrac{1}{4} - 8 = u \sin \alpha \,.\, t - \frac{1}{2} g t^2,$$

i.e.

$$-4\tfrac{3}{4} = 39 \tan \alpha - 16 \left(\frac{39}{u \cos \alpha}\right)^2 \quad \ldots\ldots \quad \ldots\ldots (1).$$

Similarly, up to the time of reaching the service-line,

$$-8 = 60 \tan \alpha - 16 \left(\frac{60}{u \cos \alpha}\right)^2 \ldots\ldots \quad \ldots\ldots (2).$$

Thus we have two equations for $u \cos \alpha$ and $\tan \alpha$.

Eliminating $\tan \alpha$, we have

$$8 \times 39 - 19 \times 15 = 16 \times 60 \times 39\,(60 - 39) \left(\frac{1}{u \cos \alpha}\right)^2.$$

$$\therefore\ u \cos \alpha = 171 \text{ feet per second, nearly.}$$

Also, from (1), we have

$$39 \tan \alpha = \frac{39 \times 3}{20 \times 7} - \frac{19}{4} = -\frac{39}{10}, \text{ nearly.}$$

$$\therefore\ \tan \alpha = -\frac{1}{10}, \text{ nearly,}$$

i.e. $\alpha = 5^\circ 43'$ below the horizontal.

3. The velocity on leaving the plane

$$= \sqrt{(16)^2 - 2g \,.\, 6 \sin 30^\circ} = 8 \text{ feet per second,}$$

at an elevation of 30°; therefore the vertical velocity then

$$= 8 \sin 30^\circ = 4 \text{ feet per second;}$$

hence the greatest height attained

$$= \frac{4^2}{2g} = \frac{1}{4} \text{ foot above the top of the plane,}$$

i.e.

$$6 \sin 30^\circ + \frac{1}{4},$$

i.e.

$$3\tfrac{1}{4} \text{ feet above the foot of the plane.}$$

Also, if R be the required range, and T be the time of flight from the top of the plane, the horizontal velocity being $8\cos 30°$, we have

$$R=6\cos 30°+8\cos 30°\,.\,T=\sqrt{3}\,(3+4T);$$

and in time T the particle descends a distance of ($6\sin 30°$) feet, *i.e.* 3 feet, below the top of the plane; hence

$$-3=8\sin 30°\,.\,T-\frac{1}{2}gT^2,\ \textit{i.e.}\ 16T^2-4T-3=0,$$

whence $$T=\frac{1+\sqrt{13}}{8}.$$

$$\therefore\ R=\frac{\sqrt{3}}{2}(7+\sqrt{13})=9{\cdot}185\ldots\text{ feet.}$$

4. Whilst in the sling, the stone in 2 seconds describes a distance of ($21\times 2\pi\times 3$) feet; therefore its velocity when it leaves the sling $=63\pi=198$ feet per second, in a horizontal direction; hence the required range$=198t$, where

$$6=\frac{1}{2}gt^2,\ \text{so that}\ t=\frac{\sqrt{6}}{4}\ \text{second};$$

thus the range

$$=198\times\frac{\sqrt{6}}{4}=121\text{ feet, nearly.}$$

5. If there were no gravity the two charges would meet, since the guns are pointing straight at one another. But the only effect of gravity is to pull each charge through the same vertical distance, so that they meet at a point vertically below where they would have met had there been no gravity.

Also the time is the same as if there had been no gravity and

$$=\frac{100}{1100+900}=\frac{1}{20}\text{ sec.}$$

The horizontal distance of the meeting point from lower gun

$$=1100\cos 30°\times\frac{1}{20}=47{\cdot}63\text{ feet.}$$

The vertical distance

$$=1100\sin 30°\,.\,\frac{1}{20}-\frac{1}{2}\,.\,g\,.\,\frac{1}{20^2}=27{\cdot}46\text{ feet.}$$

6. If u be the velocity of projection, θ be the proper elevation, and R be the distance of the mark supposed on the same level, then

$$R=\frac{2}{g}\times u\cos\theta\times u\sin\theta=\frac{u^2}{g}\sin 2\theta.$$

Also $$R-a=\frac{u^2}{g}\sin 2\alpha,\ \text{and}\ R+b=\frac{u^2}{g}\sin 2\beta.$$

Hence $$a+b=\frac{u^2}{g}(\sin 2\beta-\sin 2\alpha),$$

and $$R(\sin 2\beta-\sin 2\alpha)=a\sin 2\beta+b\sin 2\alpha.$$

$$\therefore\ \sin 2\theta=R\div\frac{u^2}{g}=\frac{a\sin 2\beta+b\sin 2\alpha}{\sin 2\beta-\sin 2\alpha}\cdot\frac{\sin 2\beta-\sin 2\alpha}{a+b}.$$

$$\therefore\ \sin 2\theta=\frac{a\sin 2\beta+b\sin 2\alpha}{a+b},\quad i.e.\ \ \theta=\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta+b\sin 2\alpha}{a+b}.$$

7. Up the hill the range $=\dfrac{2u^2}{g}\cdot\dfrac{\cos 45^\circ\sin 15^\circ}{\cos^2 30^\circ}$.

Down the hill the range is that with elevation 45° on a hill whose inclination to the horizon is -30°, and therefore

$$=\frac{2u^2}{g}\cdot\frac{\cos 45^\circ\sin 75^\circ}{\cos^2 30^\circ}.$$

Hence the ratio $=\dfrac{\sin 75^\circ}{\sin 15^\circ}=\tan 75^\circ=2+\sqrt{3}=3\tfrac{3}{4}$ nearly.

8. We have

$$\frac{2u^2}{g}\cdot\frac{\cos 60^\circ\sin(60^\circ-\beta)}{\cos^2\beta}=\frac{1}{2}g\cdot\left[\frac{2u}{g}\cdot\frac{\sin(60^\circ-\beta)}{\cos\beta}\right]^2.$$

Hence $\sin(60^\circ-\beta)=\cos 60^\circ$, so that $\beta=30^\circ$.

9. The ranges are $\dfrac{2u^2}{g}\cdot\dfrac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta}$,

and $$\frac{2u^2}{g}\cdot\frac{\cos(90^\circ-\alpha)\sin(90^\circ-\alpha+\beta)}{\cos^2\beta}.$$

Their difference $=\dfrac{2u^2}{g\cos^2\beta}[\sin\alpha\cos(\alpha-\beta)-\cos\alpha\sin(\alpha-\beta)]$

$$=\frac{2u^2\sin\beta}{g\cos^2\beta},$$

which is constant for all values of α.

10. The greatest range in the direction joining the point of projection to the top of the hill

$$=\frac{2u^2}{g}\cdot\frac{\cos\left(45^\circ+\frac{\beta}{2}\right)\sin\left(45^\circ-\frac{\beta}{2}\right)}{\cos^2\beta}=\frac{u^2}{g}\cdot\frac{\sin 90^\circ-\sin\beta}{\cos^2\beta}$$

$$=\frac{u^2}{g}\cdot\frac{1}{1+\sin\beta},$$

and this must at least be equal to $\dfrac{h}{\sin\beta}$. Hence the least value of u must be given by

$$\frac{u^2}{g}\cdot\frac{1}{1+\sin\beta}=\frac{h}{\sin\beta},\quad i.e.\ \ u=\sqrt{gh(1+\operatorname{cosec}\beta)}.$$

11. The greatest range $=\dfrac{u^2}{g}\cdot\dfrac{1}{1+\sin\beta}$.

Also the corresponding time of flight

$$=\frac{2u}{g}\cdot\frac{\sin\left(45^\circ+\frac{\beta}{2}-\beta\right)}{\cos\beta}=\frac{2u}{g}\cdot\frac{\sin\left(45^\circ-\frac{\beta}{2}\right)}{\cos\beta}.$$

The distance through which a particle would fall in this time

$$=\frac{1}{2}g\cdot\frac{4u^2}{g^2}\cdot\frac{\sin^2\left(45^\circ-\frac{\beta}{2}\right)}{\cos^2\beta}$$

$$=\frac{u^2}{g}\cdot\frac{1-\cos(90^\circ-\beta)}{\cos^2\beta}=\frac{u^2}{g}\cdot\frac{1-\sin\beta}{\cos^2\beta}=\frac{u^2}{g}\cdot\frac{1}{1+\sin\beta}.$$

12. If α be the angle of projection, we have (p. 132, Ex 3)

$$\cot\beta=2\tan(\alpha-\beta).$$

Hence $$\tan\alpha=\frac{1+2\tan^2\beta}{\tan\beta}.$$

This gives $$\frac{\sin\alpha}{1+\sin^2\beta}=\frac{\cos\alpha}{\sin\beta\cos\beta}=\frac{1}{\sqrt{1+3\sin^2\beta}},$$

so that $$\sin(\alpha-\beta)=\frac{\cos\beta}{\sqrt{1+3\sin^2\beta}}.$$

Hence the height of the point struck

$$=\frac{2u^2}{g}\cdot\frac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta}\sin\beta=\frac{2u^2}{g}\cdot\frac{\sin^2\beta}{1+3\sin^2\beta}.$$

The time of flight

$$=\frac{2u}{g}\cdot\frac{\sin(\alpha-\beta)}{\cos\beta}=\frac{2u}{g\sqrt{1+3\sin^2\beta}}.$$

Also what would be the horizontal range

$$=\frac{2u^2}{g}\cdot\sin\alpha\cos\alpha=\frac{u^2\sin 2\beta}{g}\cdot\frac{1+\sin^2\beta}{1+3\sin^2\beta},$$

13. Four times the square in the number of seconds

$$=4\times\left(\frac{2v\sin\alpha}{g}\right)^2=\frac{16v^2\sin^2\alpha}{g^2}=\frac{v^2\sin^2\alpha}{2g}$$

$=$ the greatest height in the trajectory.

14. If u be the initial velocity, and the projectile be at a height $h\sin^2\alpha$ at time t_1, and again at time t_2, we have

$$u^2\sin^2\alpha=2gh;$$

also t_1 and t_2 are the roots of the equation

$$h \sin^2 \alpha = u \sin \alpha \,.\, t - \frac{1}{2} g t^2,$$

i.e.
$$g t^2 - 2u \sin \alpha \,.\, t + 2h \sin^2 \alpha = 0\,;$$

hence the required time

$$= t_2 - t_1 = \sqrt{(t_1 + t_2)^2 - 4 t_1 t_2} = \sqrt{\frac{4u^2 \sin^2 \alpha}{g^2} - 8\frac{h}{g}\sin^2 \alpha}$$

$$= \sqrt{8\frac{h}{g} - 8\frac{h}{g}\sin^2 \alpha} = 2\sqrt{\frac{2h}{g}}\cos \alpha.$$

15. The attraction of the earth will draw both the bullet and the body through the same distance in the same time. The rifle must therefore be pointed at the balloon. The horizontal distance of the balloon is ($660 \cot 30°$) feet, and the horizontal velocity of the bullet is ($1320 \cos 30°$) feet per second. Hence the time that elapses before the bullet hits the body

$$= \frac{660 \cot 30°}{1320 \cos 30°} = 1 \text{ second.}$$

In this time the body will have fallen through

$$\frac{1}{2} g t^2, \textit{ i.e. } 16 \text{ feet.}$$

16. At the end of time $\frac{2V}{3g}$ seconds, the horizontal and vertical velocities of the second particle are

$$\frac{2V}{\sqrt{3}}\cos 60°, \text{ and } \frac{2V}{\sqrt{3}}\sin 60° - g\,.\,\frac{2V}{3g}, \textit{ i.e. } \frac{V}{\sqrt{3}} \text{ and } \frac{V}{3}.$$

Hence its velocity then is $\frac{2V}{3}$ at an angle of 30° to the horizon. The velocity of the first particle at this instant

$$= V - g \sin 30° \times \frac{2V}{3g} = \frac{2V}{3}.$$

The particles at this instant are therefore relatively at rest.

17. If u be the velocity of the carriage, the initial velocity of the dust is $2u$, and hence the initial velocity of the dust relative to the front wheel is u. Hence, if t be the time of flight, we have ut = horizontal distance between the centres of the wheels

$$= \sqrt{c^2 - (a-b)^2}.$$

Also
$$\frac{1}{2} g t^2 = \text{the vertical distance described} = 2(b-a).$$

$$\therefore\; c^2 - (a-b)^2 = u^2 t^2 = \frac{4u^2}{g}(b-a),$$

so that
$$u = \sqrt{\frac{(c+b-a)(c+a-b)g}{4(b-a)}}.$$

18. If v be the requisite velocity, we have

$$9000=\frac{v^2 \sin^2 30^\circ}{g}=\frac{v^2}{64}.$$

Hence, if P be the required quantity of powder, we have, since the kinetic energy generated will be proportional to the powder consumed,

$$P : 10=\frac{1}{2} mv^2 : \frac{1}{2} m (1600)^2.$$

$$\therefore\ P=\frac{10v^2}{(1600)^2}=\frac{64\times 90000}{2560000}=2\tfrac{1}{4} \text{ lbs.}$$

19. The initial horizontal velocity of the centre of inertia

$$=\frac{2\times 20 \cos 60^\circ+3\times 40 \cos 30^\circ}{2+3}=4\,(3\sqrt{3}+1).$$

The initial vertical velocity

$$=\frac{2\times 20 \sin 60^\circ+3\times 40 \sin 30^\circ}{2+3}=4\,(\sqrt{3}+3).$$

Also the acceleration of the centre of inertia is always g.

The required height therefore

$$=\frac{\{4\,(3+\sqrt{3})\}^2}{2g}=\frac{(3+\sqrt{3})^2}{4}=5\cdot 6 \text{ ft.}$$

The required distance

$$=\frac{2\times[4\,(3\sqrt{3}+1)]\,[4\,(\sqrt{3}+3)]}{32}=12+10\sqrt{3}=29\ 32 \text{ feet.}$$

20. 45 miles per hour $=66$ feet per second; and the latus rectum $=\frac{2}{g}\cdot$ (horizontal velocity)^2. Hence if the ball be thrown up vertically, we have the latus rectum

$$=\frac{2}{g}\times(66)^2=272\tfrac{1}{4} \text{ feet.}$$

(1) The horizontal velocity

$$=66+12 \cos 60^\circ=72 \text{ feet per second;}$$

so that the latus rectum $=\frac{2}{g}\times(72)^2=324$ feet.

(2) The horizontal velocity

$$=66-12 \cos 60^\circ=60 \text{ feet per second;}$$

so that the latus rectum $=\frac{2}{g}\times(60)^2=225$ feet.

21. The ends of the latus rectum are (Art. 113, Cor. II.) at a height $-\frac{u^2}{2g}\cos 2\alpha$.

Hence the required times are given by

$$-\frac{u^2}{2g}\cos 2\alpha = u\sin\alpha \,.\, t - \frac{1}{2}gt^2,$$

i.e. by
$$t^2 - \frac{2u\sin\alpha}{g}\,.\,t + \frac{u^2}{g^2}\sin^2\alpha = \frac{u^2}{g^2}\cos^2\alpha.$$

Hence
$$t = \frac{u}{g}(\sin\alpha \pm \cos\alpha).$$

22. Let l be the length of the tube, and u and u' be the velocities of the particle on entering and leaving the tube, so that

$$u'^2 = u^2 - 2g\,.\,l\sin 45^\circ = u^2 - \sqrt{2}\,.\,gl.$$

The required difference in the latera recta

$$= 2\frac{u^2\cos^2 45^\circ}{g} - 2\frac{u'^2\cos^2 45^\circ}{g} = \frac{u^2 - u'^2}{g} = \sqrt{2}\,.\,l.$$

23. Since the particle is projected horizontally, the focus is at the foot of the tower, and therefore the directrix is at a height of 100 feet above the top of the tower.

Hence $u = \sqrt{2g \times 100} = 80$ feet per second.

24. Let P be the point of projection, and Q and R be the highest points of the walls. Since

$$u = 2\sqrt{ga} = \sqrt{2g\,.\,2a},$$

the height of the directrix above P is $2a$. Hence the distances of Q and R from the directrix are each a. If S be the middle point of QR, we have $SQ = SR =$ the distance of both Q and R from the directrix $= a$. Hence S is the focus of the path, and the latus rectum is equal to QR, *i.e.* $2a$. The distance of the vertex of the path from the directrix therefore equals $\frac{a}{2}$.

Hence the constant horizontal velocity

$$= \sqrt{2g\,.\,\frac{a}{2}} = \sqrt{ga}.$$

Therefore the required time

$$= \frac{2a}{\sqrt{ga}} = 2\sqrt{\frac{a}{g}}.$$

25. Let P and Q be the two points through which the trajectory is to pass, and let S be the focus of one of the paths. Draw PK and QK' vertically to meet the directrix of the paths in K and K', and draw QM horizontally to meet PK in M. Then

$$SP - SQ = PK - QK' = PM$$

= the constant vertical distance between P and Q. Hence the difference of the distances of S from P and Q is constant, so that the locus of S is a hyperbola whose foci are P and Q.

26. If P' be the second point of the trajectory which is at the same height as P above the horizontal, the time from the starting point to P' is equal to t'. Hence, if h be the vertical height of P, then t and t' are the roots of the equation

$$h = v \sin\theta \, . \, T - \frac{1}{2} g T^2,$$

so that $$tt' = \frac{2h}{g}, \text{ and hence } h = \frac{1}{2} g t t'.$$

27. Consider the motions in directions along and perpendicular to the direction of motion at the given point. The acceleration in this direction is $-g \sin\theta$, and hence the velocity in this direction is destroyed in time $\frac{u}{g \sin\theta}$, so that at the end of this time the particle is moving at right angles to its original direction.

EXAMPLES. XX. (Page 148.)

1. The velocity with which the marble strikes the floor $= \sqrt{2g \, . \, 9}$ feet per second; therefore the velocity of rebound $= \cdot 9 \times \sqrt{2g \, . \, 9}$ feet per second. Hence the required height h is given by

$$2gh = \left(\frac{9}{10}\right)^2 \times 2g \times 9,$$

whence $$h = 7\cdot 29 \text{ feet}.$$

2. Let e be the coefficient of restitution. The velocity of the ball when it reaches the slab $= \sqrt{2g \, . \, 25}$ feet per second; hence

$$2g \times 16 = e^2 \times 2g \times 25,$$

whence $$e = \frac{4}{5} = \cdot 8.$$

3. If h be the height of the room, the velocity u when the ball hits the floor $= \sqrt{2gh}$. The velocity after the second impact $= e^2 u$, and since the ball then rises to a height $\frac{h}{2}$, we have

$$e^2 u = \sqrt{2g \times \frac{h}{2}} = \frac{u}{\sqrt{2}}, \text{ so that } e = \sqrt[4]{\frac{1}{2}}.$$

4. If v be the velocity of projection, the ball reaches the opposite wall with the velocity v unaltered, and the velocity of rebound

$$=ev=\frac{1}{2}v,$$

i.e. the ball takes twice as long in returning as it took in going.

5. The ball reaches the ceiling with the velocity equal to

$$\sqrt{(32\sqrt{3})^2-2g\,.\,16}=32\sqrt{2} \text{ feet per second};$$

and the velocity of rebound from the ceiling

$$=\frac{1}{\sqrt{2}}\times 32\sqrt{2},$$

i.e. 32, feet per second; therefore the velocity with which the ball reaches the floor

$$=\sqrt{(32)^2+2g\,.\,16}=32\sqrt{2} \text{ feet per second};$$

therefore the velocity of the rebound upwards

$$=\frac{1}{\sqrt{2}}\times 32\sqrt{2},$$

i.e. 32, feet per second; hence the height then attained by the ball

$$=\frac{(32)^2}{2g}=16 \text{ feet},$$

i.e. the ball just reaches the ceiling again.

6. If the ball move at an angle θ to the plane after the impact with velocity v, then

$$v\cos\theta=8\cos 30^\circ=4\sqrt{3},$$

and $$v\sin\theta=\frac{1}{2}\,.\,8\sin 30^\circ=2.$$

Hence, by squaring and adding, $v^2=52$, so that

$$v=7{\cdot}2 \text{ feet per second}.$$

Also, by division, $$\tan\theta=\frac{2}{4\sqrt{3}}=\frac{\sqrt{3}}{6},$$

so that, by the tables, $\theta=16^\circ\,6'$, nearly.

7. If v be the required velocity at an angle θ to the plane, and $a=36^\circ\,52'$, then

$$v\cos\theta=5\cos a=5\times\frac{4}{5}=4,$$

and $$v\sin\theta=\frac{2}{3}\times 5\sin a=2.$$

Hence, by squaring and adding, $v^2=20$, so that $v=4{\cdot}47$ feet per second. Also

$$\tan\theta=\frac{2}{4}=\frac{1}{2},\ \textit{i.e.}\ \ \theta=26^\circ\,34',$$

nearly.

8. If the plane were at an angle α to the horizon, and v the velocity after impact in direction making an angle θ with the plane, then the velocity on striking the plane

$$=\sqrt{2g\,.\,16}=32 \text{ feet per second.}$$

$$\therefore\ v\cos\theta=32\sin\alpha \quad\text{and}\quad v\sin\theta=\frac{3}{4}\times 32\cos\alpha.$$

$$\therefore\ v=\sqrt{(32\sin\alpha)^2+(24\cos\alpha)^2}=8\sqrt{9+7\sin^2\alpha};$$

also
$$\tan\theta=\frac{3}{4}\cot\alpha.$$

Hence (1) $v=8\sqrt{9+\frac{7}{4}}=26{\cdot}2$ feet per second;

and
$$\tan\theta=\frac{3}{4}\times\sqrt{3},\ \textit{i.e.}\ \ \theta=52^\circ\,25'.$$

(2) $v=8\sqrt{9+\frac{7}{2}}=20\sqrt{2}=28{\cdot}3$ feet per second;

and
$$\tan\theta=\frac{3}{4},\ \textit{i.e.}\ \ \theta=36^\circ\,52'.$$

(3) $v=8\sqrt{9+\frac{21}{4}}=30{\cdot}2$ feet per second;

and
$$\tan\theta=\frac{3}{4}\cdot\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{4},\ \textit{i.e.}\ \ \theta=23^\circ\,25'.$$

EXAMPLES. XXI. (Pages 154–156.)

1. Let v and v' be the required velocities. Since the total momentum is unaltered,

$$\therefore\ 4v+3v'=4\times5+3\times4=32.$$

By Newton's Law,

$$v-v'=-\frac{1}{2}(5-4)=-\frac{1}{2}.$$

Hence, solving, $v=4\frac{5}{14}$ feet per second, and $v'=4\frac{6}{7}$ feet per second.

2. Here $$10v+8v'=10\times6+8\times3=84,$$

and
$$v-v'=-\frac{3}{4}(6-3)=-\frac{9}{4};$$

hence $v=3\frac{2}{3}$ feet per second, and $v'=5\frac{11}{12}$ feet per second.

3. If v and v' be the required velocities, and m be the mass of either sphere, we have

$$mv+mv'=m\times 12+m\times(-6),$$

i.e.
$$v+v'=6;$$

and
$$v-v'=-\frac{1}{3}[12-(-6)]=-6;$$

hence $v=0$, and $v'=6$ feet per second; thus the first sphere comes to rest, and the second turns back with a velocity of 6 feet per second.

4. The equations determining the impact are

$$mv+2mv'=mu+2m\cdot\frac{u}{7}=m\cdot\frac{9u}{7},$$

and
$$v-v'=-\frac{3}{4}\left(u-\frac{u}{7}\right)=-\frac{9u}{14}.$$

Solving, we have $\quad v'=\dfrac{9u}{14}$, and $v=0$;

so that the first ball remains at rest.

5. The equations are

$$2mv+mv'=2m\cdot u+m(-2u)=0,$$

and
$$v-v'=-\frac{5}{6}[u-(-2u)]=-\frac{5u}{2}.$$

Hence, solving, we have

$$v=-\frac{5u}{6},\text{ and } v'=\frac{10u}{6}=\frac{5}{6}\cdot 2u,$$

so that each ball turns back with $\frac{5}{6}$ths of its original velocity.

6. If v and v' be the velocities after impact, m be the mass of each sphere, and u be the velocity of the first sphere before impact, then we have

$$mv+mv'=mu,\ i.e.\ v+v'=u,$$

and
$$v-v'=-e(u-0)=-eu.$$

Hence, by eliminating u,

$$v-v'=-e(v+v');$$

hence
$$v(1+e)=v'(1-e),\ i.e.\ v:v'=1-e:1+e.$$

7. If v and v' be the velocities after impact, then

$$mv+emv'=mu-em\cdot eu,$$

i.e.
$$v+ev'=u(1-e^2);$$

and
$$v-v'=-e(u+eu)=-eu(1+e);$$

$$\therefore\ v'(1+e)=u(1-e^2)+eu(1+e);$$

so that
$$v'=u(1-e+e)=u,\text{ and } v=u(1-e-e^2).$$

8. Here $$2v+u=2u,$$
and $$v-u=-eu.$$
Hence $$u=2v, \text{ and } e=\frac{1}{2}.$$

9. We have $$m\cdot\frac{3}{5}u+m'v=mu,$$
and $$\frac{3}{5}u-v=-\frac{3}{5}u.$$
Hence $$v=\frac{6}{5}u,$$
so that the ratio of the velocities after the impact is 1 : 2.

Also, from the first equation, $m=3m'$.

10. If v and v' be the velocities of the first and second spheres respectively after impact, then
$$2v+6v'=2\times 12+6\times 4=48,$$
i.e. $$v+3v'=24;$$
and $$v-v'=-(12-4)=-8;$$
hence $$v=0, \text{ and } v'=8 \text{ feet per second,}$$
i.e. the first sphere is brought to rest, and the second goes on with a velocity of 8 feet per second. Now let V and V' be the velocities of the second and third spheres respectively after impact; then
$$6V+12V'=6\times 8+12\times 2=72,$$
i.e. $$V+2V'=12;$$
and $$V-V'=-(8-2)=-6;$$
hence $$V=0, \text{ and } V'=6 \text{ feet per second,}$$
i.e. the second sphere is brought to rest, and the third goes on with a velocity of 6 feet per second.

11. If the balls meet after t seconds, the first has fallen a distance
$$\frac{1}{2}gt^2,$$
and the second has ascended a distance
$$128t-\frac{1}{2}gt^2;$$
hence $$\frac{1}{2}gt^2+128t-\frac{1}{2}gt^2=64, \text{ so that } t=\frac{1}{2} \text{ second,}$$
and thus $$\frac{1}{2}gt^2=4 \text{ feet,}$$

i.e. impact takes place at a height of 60 feet above the ground. Also the velocities are gt, *i.e.* 16 feet per second, downwards, and $128-gt$, *i.e.* 112 feet per second, upwards. Hence, if v and v' be the velocities upwards after impact, we have $v+v'=-16+112=96$,

and $$v-v'=-\frac{1}{2}(-16-112)=64;$$

whence $v=80$ feet per second, and $v'=16$ feet per second.

Also the required times t_1 seconds and t_2 seconds are given by

$$-60=80t_1-\frac{1}{2}gt_1^2$$

and $$-60=16t_2-\frac{1}{2}gt_2^2;$$

whence $t_1=5{\cdot}66$ seconds and $t_2=2{\cdot}5$ seconds.

12. If u be the velocity of the first sphere before impact, v and v' be the velocities of the two spheres after impact, and v make an angle θ with the line of centres, then the impulse being along this line, the velocity perpendicular to it is unaltered, so that $v\sin\theta=u\sin 30°$.

Also, the spheres being inelastic ($e=0$), their velocities along the line of centres are equal, so that

$$v\cos\theta=v'.$$

Also the momentum along the line of centres is unaltered; hence

$$1\,.\,u\cos 30°=1\,.\,v\cos\theta+2v'=3v\cos\theta.$$

$$\therefore\ \frac{1}{3}\tan\theta=\tan 30°=\frac{1}{\sqrt{3}},\ \textit{i.e.}\ \theta=60°,$$

so that the direction of motion is turned through an angle equal to

$$60°-30°,\ \textit{i.e.}\ 30°.$$

13. If u be the common velocity before impact, and v and v' be the velocities after impact at angles θ and ϕ to the line of centres, then, for the motion perpendicular to this line, we have

$$v\sin\theta=u\sin 30°=\frac{u}{2},\ \text{and}\ v'\sin\phi=u\sin 60°=\frac{u\sqrt{3}}{2}.$$

Also, the momentum along the line of centres being unaltered,

$$v\cos\theta+v'\cos\phi=u(\cos 30°+\cos 60°),$$

and also $$v\cos\theta-v'\cos\phi=-u(\cos 30°-\cos 60°).$$

Therefore

$$v\cos\theta=u\cos 60°=\frac{u}{2},\ \text{and}\ v'\cos\phi=u\cos 30°=\frac{u\sqrt{3}}{2};$$

hence $$\tan\theta=\frac{u}{2}\div\frac{u}{2}=1,\ \textit{i.e.}\ \theta=45°,$$

and $$\tan\phi=\frac{u\sqrt{3}}{2}\div\frac{u\sqrt{3}}{2}=1,\ \textit{i.e.}\ \phi=45°.$$

14. If u be the velocity before impact, and v and v' be the velocities after impact at angles θ and ϕ respectively to the line of centres, then

$$v \sin\theta = u \sin 30° = \frac{u}{2}, \text{ and } v' \sin\phi = u \sin 90° = u;$$

also

$$v \cos\theta + v' \cos\phi = u \cos 30°, \text{ and } v \cos\theta - v' \cos\phi = -\frac{1}{3} u \cos 30°.$$

$$\therefore v \cos\theta = \frac{2}{3} u \cos 30° = \frac{u}{\sqrt{3}}, \text{ and } v' \cos\phi = \frac{2u}{\sqrt{3}};$$

hence

$$\tan\theta = \frac{u}{2} \div \frac{u}{\sqrt{3}} = \frac{\sqrt{3}}{2}, \text{ and } \tan\phi = u \div \frac{2u}{\sqrt{3}} = \frac{\sqrt{3}}{2} = \tan\theta,$$

so that $\theta = \phi$.

Also, since $v' \cos\phi = 2v \cos\theta$, we have $v' = 2v$.

15. If the velocities before impact be u and u', at angles α and $\frac{\pi}{2} + \alpha$ to the line of centres, and if they be v and v' at angles θ and ϕ to that line after impact, then

$$v \sin\theta = u \sin\alpha \quad \text{...(1)},$$

and

$$v' \sin\phi = u' \cos\alpha \quad \text{...(2)};$$

also

$$v \cos\theta + v' \cos\phi = u \cos\alpha - u' \sin\alpha \quad \text{...(3)},$$

and

$$v \cos\theta - v' \cos\phi = -u \cos\alpha - u' \sin\alpha \quad \text{...(4)};$$

adding and subtracting (3) and (4), we have

$$v \cos\theta = -u' \sin\alpha, \text{ and } v' \cos\phi = u \cos\alpha;$$

hence, from (1) and (2), we have

$$\tan\theta = -\frac{u}{u'}, \text{ and } \tan\phi = \frac{u'}{u} = -\tan\theta,$$

i.e. $\theta = \frac{\pi}{2} + \phi.$

16. If v and v' be the velocities after impact at angles θ and ϕ to the line of centres, then

$$v \sin\theta = u\sqrt{3} \sin 30° = \frac{u\sqrt{3}}{2}, \text{ and } v' \sin\phi = u \sin 60° = \frac{u\sqrt{3}}{2};$$

also

$$v \cos\theta + v' \cos\phi = u\sqrt{3} \cos 30° + u \cos 60° = 2u,$$

and

$$v \cos\theta - v' \cos\phi = -(u\sqrt{3} \cos 30° - u \cos 60°) = -u.$$

$$\therefore v \cos\theta = \frac{u}{2}, \text{ and } v' \cos\phi = \frac{3u}{u};$$

hence

$$\tan\theta = \frac{u\sqrt{3}}{2} \div \frac{u}{2} = \sqrt{3}, \textit{ i.e. } \theta = 60°,$$

and

$$\tan\phi = \frac{u\sqrt{3}}{2} \div \frac{3u}{2} = \frac{1}{\sqrt{3}}, \textit{ i.e. } \phi = 30°.$$

17. If v and v' be the velocities after impact at angles θ and ϕ to the line of centres, and

$$\sin \alpha = \frac{5}{13}, \quad \sin \beta = \frac{3}{5},$$

we have

$$v \sin \theta = 13u \sin \alpha = 5u \quad \ldots\ldots \quad (1),$$

and

$$v' \sin \phi = 5u \sin \beta = 3u \quad \ldots\ldots \quad (2);$$

also

$$v \cos \theta - v' \cos \phi = -\frac{1}{2}(13u \cos \alpha - 5u \cos \beta)$$

$$= -\frac{1}{2}\left(13u \times \frac{12}{13} - 5u \times \frac{4}{5}\right) = -4u;$$

and $5m \,.\, v \cos \theta + m \,.\, v' \cos \phi = 5m \,.\, 13u \cos \alpha + m \,.\, 5u \cos \beta,$

or

$$5v \cos \theta + v' \cos \phi = u(60 + 4) = 64u.$$

$$\therefore \; v \cos \theta = 10u.$$

Hence, by (1),

$$v^2 = u^2[5^2 + (10)^2], \text{ so that } v = u \times 11{\cdot}180,$$

and

$$\tan \theta = \frac{5u}{10u} = \frac{1}{2}, \text{ i.e. } \theta = 26° \, 34'.$$

Also

$$v' \cos \phi = 64u - 5v \cos \theta = 14u.$$

$$\therefore \; v'^2 = u^2[3^2 + (14)^2], \text{ so that } v' = u \times 14{\cdot}318,$$

and

$$\tan \phi = \frac{3u}{14u} = \frac{3}{14}, \text{ i.e. } \theta = 12° \, 6'.$$

EXAMPLES. XXII. (Pages 162—164.)

1. If u be the velocity of projection, and v be the velocity of return, then the horizontal component $u \cos \alpha$ is by the impact reduced to $eu \cos \alpha$,

i.e.

$$v \cos \beta = eu \cos \alpha;$$

also the vertical component of the velocity is unaltered by the impact, so that

$$v \sin \beta = u \sin \alpha,$$

numerically; hence

$$\tan \beta = \frac{1}{e} \tan \alpha, \text{ i.e. } \tan \alpha = e \tan \beta.$$

2. The velocity of the sphere when it first hits the table

$$= \sqrt{2g \,.\, 16} = 32 \text{ feet per second};$$

therefore the velocity of the first rebound $= 32e$ feet per second, and the height then ascended

$$= \frac{(32e)^2}{2g} = 16e^2;$$

and so on. Hence the distance described

$$=16+2(16e^2+16e^4+\ldots\ldots)=16+\frac{32e^2}{1-e^2}$$

$$=16\cdot\frac{1+e^2}{1-e^2}=16\cdot\frac{1+\left(\frac{7}{9}\right)^2}{1-\left(\frac{7}{9}\right)^2}=65 \text{ feet.}$$

Also the time in which the velocity $32e$ is destroyed $=\frac{32e}{g}=e$ and so on. Thus the sphere will come to rest in time

$$=\sqrt{\frac{2\times 16}{g}}+2(e+e^2+\ldots)=1+\frac{2e}{1-e}=\frac{1+e}{1-e}$$

$$=\frac{9+7}{9-7}=8 \text{ seconds.}$$

3. The velocity u of the ball on first striking the plane

$$=\sqrt{2g\cdot 48}=32\sqrt{3} \text{ feet per second;}$$

the successive velocities of impact are eu, e^2u......; the successive heights ascended and again descended are

$$\frac{e^2u^2}{2g},\quad \frac{e^3u^4}{2g}\ldots\ldots;$$

hence the total distance described

$$=48+2(48e^2+48e^4+\ldots\ldots)=48+\frac{96e^2}{1-e^2}=48\cdot\frac{1+e^2}{1-e^2}$$

$$=48\cdot\frac{3^2+1}{3^2-1}=60 \text{ feet.}$$

Also the time of descending 48 feet

$$=\sqrt{\frac{2\times 48}{g}}=\sqrt{3} \text{ second,}$$

and the time in which the velocity eu is destroyed $=\frac{eu}{g}$, and so on. Thus the required time

$$=\sqrt{3}+2\left(e\cdot\frac{32\sqrt{3}}{g}+e^2\cdot\frac{32\sqrt{3}}{g}+\ldots\ldots\right)=\sqrt{3}+\frac{2\sqrt{3}\cdot e}{1-e}$$

$$=2\sqrt{3}=3{\cdot}464\ldots\ldots \text{ seconds.}$$

4. The horizontal velocity throughout

$$=64\cos 30^\circ=32\sqrt{3} \text{ feet per second;}$$

and the vertical velocities at the successive rebounds are

$$e\cdot 64\sin 30^\circ,\ \textit{i.e.}\ 32e,\ 32e^2,$$

and so on. Hence the sum of the horizontal trajectories until rebounding ceases

$$=\frac{2}{g}[(32)^2\sqrt{3}+(32)^2.\sqrt{3}.e+......]=64\sqrt{3}.\frac{1}{1-e}$$

$=443\cdot4$ feet, nearly. Also the time elapsed

$$=\frac{4\times64\sqrt{3}}{\text{horizontal velocity}}=\frac{4\times64\sqrt{3}}{32\sqrt{3}}=8 \text{ seconds.}$$

5. The velocity of the ball on hitting the plane $=2g$; therefore the component perpendicular to the plane

$$=2g\sin 60^\circ=g\sqrt{3},$$

which becomes $\frac{3}{4}g\sqrt{3}$ after the impact. Also the acceleration perpendicular to the plane

$$=g\cos 30^\circ=\frac{\sqrt{3}}{2}g.$$

Hence the required time t is given by

$$0=\frac{3}{4}g\sqrt{3}.t-\frac{1}{2}g.\frac{\sqrt{3}}{2}.t^2,$$

whence $t=3$ seconds.

6. The velocity of the ball on striking the stone

$$=\sqrt{2g.\frac{h}{2}}=\sqrt{gh};$$

and immediately afterwards the velocity is made up of $\sqrt{gh}.\frac{1}{\sqrt{2}}$ parallel to the stone, and $\sqrt{gh}.\frac{1}{\sqrt{2}}$ perpendicular to the stone upwards; therefore the velocity $=\sqrt{gh}$ horizontally. Hence, if the ball reach the ground in time t at a distance x from the foot of the tower, we have

$$\frac{h}{2}=\frac{1}{2}gt^2, \text{ whence } t=\sqrt{\frac{h}{g}},$$

and

$$x=vt=\sqrt{gh}.\sqrt{\frac{h}{g}}=h.$$

7. If v and v' be the velocities after the first collision, then

$$v+v'=3, \text{ and } v-v'=-\frac{1}{3}(3-0)=-1,$$

so that $v'=2$ feet per second, and the second ball reaches the wall in $\frac{3}{2}$ second; v' then becomes $\frac{1}{3}\times2$ in the opposite direction; also since

$$v=\frac{1}{2}\times2=1,$$

the first ball has moved $\frac{3}{2}$ feet towards the wall, and is, therefore, $\frac{3}{2}$ feet from the second ball; and the relative velocity of the two balls

$$=v+\frac{1}{3}v'=1+\frac{2}{3}=\frac{5}{3};$$

therefore the additional time before they meet

$$=\frac{3}{2}\div\frac{5}{3}=\frac{9}{10}\text{ second.}$$

Hence the whole time between the first and second collisions between them

$$=1\tfrac{1}{2}+\frac{9}{10}=2{\cdot}4\text{ seconds.}$$

8. If u be the velocity of A originally and r be the radius of the groove, the time $t=\frac{\pi r}{u}$. After the impact the relative velocity $=-eu$. Hence the time that elapses before the second impact

$$=\frac{2\pi r}{eu}=\frac{2t}{e}.$$

9. If the marbles be projected from A with equal velocities u, they will meet at B the other end of the diameter through A; and if v and v' be their velocities measured in the same direction as that of the first marble after the first impact, we have

$$10m\,.\,v+11m\,.\,v'=10m\,.\,u-11m\,.\,u,$$

i.e.
$$10v+11v'=-u;$$

also
$$v-v'=-\frac{3}{4}[u-(-u)]=-\frac{3u}{2};$$

whence
$$v=-\frac{5}{6}u,\text{ and }v'=\frac{2u}{3}.$$

Therefore each marble returns, and the portions of the whole circumference which they describe before meeting again are as $v:v'$, *i.e.* as $5:4$. Hence the marble of mass $10m$ will describe a distance equal to $\frac{5}{9}$ths of the circumference from B, *i.e.* to a point which is $\frac{1}{18}$th of the circumference beyond A.

10. The equations for the impact are (since the sphere M necessarily proceeds to move along the line of centres and therefore ϕ is zero)

$$eMv\cos\theta+Mv'=eM\,.\,u\cos\alpha,$$

and
$$v\cos\theta-v'=-eu\cos\alpha.$$

Hence $(1+e)v\cos\theta=0$, so that θ is a right angle.

Hence, after the impact, the spheres are moving in perpendicular directions.

11. Here we have $\alpha=30°$, and $\theta=60°$, so that

$$mv\cos 60°+mv'=mu\cos 30° \quad\text{.................(1)},$$

$$v\cos 60°-v'=-e\,.\,u\cos 30° \quad\text{..............(2)},$$

and $$v\sin 60°=u\sin 30° \quad\text{....................(3)}.$$

From the first two equations, we have

$$2v\cos 60°=(1-e)\cos 30°.$$

Hence, from the third equation,

$$2\cot 60°=(1-e)\cot 30°,$$

i.e. $$e=\frac{1}{3}.$$

12. The equations are

$$mv+mv'\cos\theta=mu \quad\text{....................(1)},$$

$$v-v'\cos\theta=-eu \quad\text{.....................(2)},$$

and $$v'\sin\theta=u \quad\text{....................(3)}.$$

From the first two equations, we have

$$2v'\cos\theta=(1+e)u.$$

Hence, from the third equation,

$$2\cot\theta=(1+e).$$

Hence the required angle

$$=\frac{\pi}{2}-\theta=\frac{\pi}{2}-\cot^{-1}\frac{1+e}{2}=\tan^{-1}\frac{1+e}{2}.$$

13. The equations are

$$mv\cos\theta+mv\ \cos\phi=mu\cos\alpha+m(-u\cos\alpha)=0 \quad\text{.........(1)},$$

$$v\cos\theta-v'\cos\phi=-e[u\cos\alpha-(-u\cos\alpha)]=-2eu\cos\alpha \quad\text{......(2)},$$

$$v\sin\theta=u\sin\alpha \quad\text{..............................(3)},$$

$$v'\sin\phi=u\sin\alpha \quad\text{..........................(4)},$$

and $$\alpha=\tan^{-1}\sqrt{e} \quad\text{.........................(5)}.$$

From (1) and (2),

$$v\cos\theta=-eu\cos\alpha=-v'\cos\phi.$$

Hence, from (3) and (4),

$$\tan\theta=-\tan\phi=-\frac{1}{e}\tan\alpha=-\frac{\tan\alpha}{\tan^2\alpha}=-\cot\alpha.$$

Hence $\theta=\frac{\pi}{2}+\alpha$, and $\phi=\frac{\pi}{2}-\alpha$.

Therefore the directions of motion are both turned through a right angle.

14. Let u be the initial velocity of the striking ball, v be its velocity after the impact, and V be the velocity of each of the two balls in directions inclined respectively at $30°$ to the direction of u. We then have $mu = mv + 2mV\cos 30° = mv + mV\sqrt{3}$..............(1),

$$-eu\cos 30° = v\cos 30° - V \text{.................(2).}$$

But v is zero; hence

$$u = V\sqrt{3} = \sqrt{3} \,.\, eu\cos 30° = \frac{3}{2}eu,$$

so that
$$e = \frac{2}{3}.$$

15. For the first impact, we have $mv + 2mv' = mu$,

and
$$v - v' = -eu = -\frac{2}{3}u.$$

Hence
$$3v' = \frac{5}{3}u, \text{ so that } v' = \frac{5}{9}u.$$

The velocity of the third ball after the impact, similarly,

$$= \frac{5}{9}v' = \left(\frac{5}{9}\right)^2 u,$$

and so on,

so that the velocity of the last ball $= \left(\frac{5}{9}\right)^4 u.$

16. The velocity u of the ball on hitting the horizontal plane

$$= \sqrt{2gl\sin\alpha}.$$

Immediately after the impact the velocities of the ball are $u\cos\alpha$ horizontally, and $eu\sin\alpha$ vertically. Hence the range

$$= \frac{2\,.\,u\cos\alpha\,.\,eu\sin\alpha}{g} = 4el\cos\alpha\sin^2\alpha.$$

17. The velocity u of the ball on hitting the plane is $\sqrt{2gh}$. After the impact, the velocity perpendicular to the inclined plane is

$$eu\cos 60°, \text{ i.e. } \frac{eu}{2}.$$

Also the velocity down the inclined plane then

$$= u\sin 60° = \frac{u\sqrt{3}}{2}.$$

The acceleration perpendicular to the inclined plane is $g\cos 60°$, so that the time of flight is $\frac{2eu}{g}$. Also the acceleration down the plane

$$= g\sin 60° = \frac{g\sqrt{3}}{2}.$$

Hence the required distance

$$=\frac{u\sqrt{3}}{2}\cdot\frac{2eu}{g}+\frac{1}{2}\,\frac{g\sqrt{3}}{2}\cdot\frac{4e^2u^2}{g^2}$$

$$=(e+e^2)\,\frac{u^2\sqrt{3}}{g}=2\sqrt{3}ne\,(1+e)\text{ feet.}$$

18. Since the height of the rail above the ground is half the radius, the radius from the rail to the centre makes an angle of 60° with the vertical. The velocity along this radius is destroyed, and the velocity, 16 cos 60°, perpendicular to it is unaltered. Hence, after leaving the rail, the initial velocity is 8 feet per second at an angle of 60° with the horizontal. Hence the latus rectum

$$=4\,\frac{8^2\cos^2 60^\circ}{2g}=1\text{ foot.}$$

19. Draw AM and BN perpendicular to the vertical plane, and let P be the point of impact on this plane, so that

$$\tan BPN=e\tan APM.$$

Let AP meet BN in C. Then we have $\tan BPN=e\tan CPN$, so that $BN=e\,.\,NC$. Hence the required construction.

20. Let $ABCA$ be the path of the ball, O being the centre of the circular table, and let

$$\angle OAB=\angle OBA=\theta,\quad \angle OBC=\angle OCB=\phi,$$

and $$\angle OCA=\angle OAC=\psi.$$

We have then $$\cot\phi=e\cot\theta,$$

and $$\cot\psi=e\cot\phi=e^2\cot\theta.$$

Also $$\theta+\phi+\psi=90^\circ,$$

so that

$$1=\tan\theta\tan\phi+\tan\phi\tan\psi+\tan\psi\tan\theta=\tan^2\theta\left(\frac{1}{e}+\frac{1}{e^3}+\frac{1}{e^2}\right).$$

Hence $$\tan\theta=\sqrt{\frac{e^3}{1+e+e^2}}.$$

Also, let u, v and w be the successive velocities of the ball, so that

$$v^2=u^2(\sin^2\theta+e^2\cos^2\theta),$$

and $$w^2=v^2(\sin^2\phi+e^2\cos^2\phi).$$

Hence $$\frac{w^2}{u^2}=(\sin^2\theta+e^2\cos^2\theta)(\sin^2\phi+e^2\cos^2\phi).$$

Now $$\frac{\sin^2\theta}{e^3}=\frac{\cos^2\theta}{1+e+e^2}=\frac{1}{(1+e)(1+e^2)}.$$

Also, since $$\tan\phi=\frac{1}{e}\tan\theta,$$

we have $$\frac{\sin^2\phi}{e}=\frac{\cos^2\phi}{1+e+e^2}=\frac{1}{(1+e)^2}.$$

Hence $$\frac{w^2}{u^2}=\frac{e^3+e^2(1+e+e^2)}{(1+e)(1+e^2)}\cdot\frac{e+e^2(1+e+e^2)}{(1+e)^2}$$
$$=\frac{e^2(1+e)^2 \, . \, e(1+e)(1+e^2)}{(1+e)^3(1+e^2)}.$$

Therefore $$w=u \, . \, e^{\frac{3}{2}}.$$

For the second part of the question, let $ABCDA$ be the path of the ball and let

$$\angle OAB=\angle OBA=\theta,$$
$$\angle OBC=\angle OCB=\phi,$$
$$\angle OCD=\angle ODC=\psi,$$
and $$\angle ODA=\angle OAD=\chi.$$

As above, we have
$$\cot\phi=e\cot\theta,\quad \cot\psi=e\cot\phi=e^2\cot\theta,$$
and $$\cot\chi=e\cot\psi=e^3\cot\theta.$$

Now, since $ABCD$ is a quadrilateral in a circle we have
$$\theta+\phi+\psi+\chi=\pi.$$
$$\therefore\ \tan\theta+\tan\phi+\tan\psi+\tan\chi$$
$$=\tan\theta\tan\phi\tan\psi+\tan\theta\tan\phi\tan\chi+\tan\theta\tan\psi\tan\chi$$
$$+\tan\phi\tan\psi\tan\chi.$$
$$\therefore\ \tan\theta\left(1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right)=\tan^3\theta\left(\frac{1}{e^3}+\frac{1}{e^4}+\frac{1}{e^5}+\frac{1}{e^6}\right)$$
$$=\tan^3\theta \, . \, \frac{1}{e^3}\left[1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right].$$
$$\therefore\ \tan\theta=e^{\frac{3}{2}}.$$

21. As in Ex. 19, page 140, the initial vertical and horizontal velocities of the centre of inertia may be found. The latter is unaltered by the impacts. Hence the latus rectum of the path described by the centre of inertia which, by Art. 113, Cor. I., depends only on the horizontal velocity remains unaltered. The size therefore of the parabolas, portions of which are described, remains the same, *i.e.* the arcs are all portions of the same parabola.

EXAMPLES. XXIII. (Page 172.)

1. The required tension
$$=\frac{mv^2}{r}=\frac{5\times 6^2}{3}=60 \text{ poundals.}$$

2. Let n be the required number of revolutions, so that the velocity of the mass
$$=\left(\frac{n}{60}\times 2\pi\times 4\right) \text{ feet per second.}$$

Then $9g=$ the tension of the string

$$=\frac{8}{4}\left(\frac{2n\pi}{15}\right)^2.$$

Hence $n=28\cdot6$, nearly.

3. Here $$20g=\frac{5}{5}\left(\frac{n}{60}\times2\pi\times5\right)^2;$$

whence $n=48\cdot3$, nearly, *i.e.* the greatest number of *complete* revolutions is 48.

4. If n be the required number of revolutions, the velocity of the mass

$$=\left(\frac{n}{60}\times2\pi\times2\tfrac{1}{2}\right)\text{ feet per second}=\frac{n\pi}{12}\text{ feet per second.}$$

Hence $$5g=\frac{1}{2\frac{1}{2}}\times\left(\frac{n\pi}{12}\right)^2;$$

so that $n=77$, nearly.

5. If v be the velocity of the mass m, we have

$$9m\,.\,g=\frac{mv^2}{2};$$

hence $v=24$ feet per second.

6. We have $1\,.\,g=10\,.\,\frac{v^2}{50}$, so that the required number of turns

$$=\frac{60v}{2\pi\,.\,50}=\frac{6\sqrt{5\times981}}{10\pi}=\text{about }\frac{42}{\pi}=13\cdot4.$$

7. 30 miles per hour $=44$ feet per second; hence the required force

$$=\frac{mv^2}{r}=2240\times10\times\frac{(44)^2}{600}\text{ poundals}$$

$$=\frac{10}{32}\times\frac{(44)^2}{600}\text{ tons wt.}=1\tfrac{1}{120}\text{ tons wt.}$$

8. Here the required force

$$=2240\times12\times\frac{(88)^2}{1200}\text{ poundals}$$

$$=\frac{12}{32}\times\frac{(88)^2}{1200}\text{ tons wt.}=2\cdot42\text{ tons wt.}$$

EXAMPLES. XXIV. (Pages 178 181.)

1. Let T poundals be the tension of the string, and v be the velocity of the particle; then, since the particle has no vertical acceleration, we have

$$T\cos 45^\circ = 4g, \text{ i.e. } T = 4\sqrt{2} \text{ lbs. wt.} = 5{\cdot}66 \text{ lbs. wt.}$$

Also, resolving the tension horizontally, r being the radius of the circle described by the particle, we have

$$T\sin 45^\circ = 4 \times \frac{v^2}{r};$$

hence $$g = \frac{v^2}{r} = v^2 \div \frac{3}{\sqrt{2}}, \text{ i.e. } v^2 = \frac{3g}{\sqrt{2}} = 48\sqrt{2},$$

and therefore $v = 8{\cdot}24$ feet per second, nearly.

2. Let a be the required inclination, T poundals be the tension of the string, and v be the velocity of the mass m in feet per second.

Then $$v = \frac{200}{60} \times 2\pi \times \frac{20}{12}\sin a = \frac{100\pi \,.\sin a}{9}.$$

Also $$T\cos a = mg;$$

and $$T\sin a = \frac{mv^2}{\frac{20}{12}\sin a} = \frac{12m}{20\sin a}\cdot\left(\frac{100\pi\,.\sin a}{9}\right)^2$$

$$= \frac{4m\pi^2 \,.\, 500\sin a}{27}, \text{ i.e. } T = \frac{2000}{27}m\pi^2.$$

Hence $$\cos a = \frac{mg}{T} = \frac{27}{2000}\cdot\frac{g}{\pi^2} = \frac{54}{125\pi^2},$$

i.e. $$a = \cos^{-1}\frac{54}{125\pi^2}, \text{ i.e. about } 87^\circ 30'.$$

3. Let T be the tension of the string, a be its inclination to the vertical, and v be the velocity of the mass.

Then $$v = \frac{30}{60} \times 2\pi \times 4\sin a = 4\pi\sin a.$$

Also $$T\sin a = \frac{40v^2}{4\sin a} = 160\pi^2 \,.\sin a,$$

so that $$T = 160\pi^2 \text{ poundals.}$$

Also $$T\cos a = 40g;$$

hence $$\cos a = \frac{40 \times 32}{160\pi^2} = \frac{8}{\pi^2}.$$

Hence $$a = \cos^{-1}\frac{8}{\pi^2}, \text{ i.e. about } 35^\circ 51'.$$

4. Here $$T\cos 60^\circ = mg,$$
and $$T\sin 60^\circ = \frac{mv^2}{3\sin 60^\circ}.$$

Hence $$v^2 = 3\sin 60^\circ \,.\, g\tan 60^\circ = 144,$$
so that $v = 12$ feet per second.

5. Here $v = 88$ feet per second, $r = 770$ feet, and $m = 2\times 2240$ lbs.
Hence the lateral pressure on the rails
$$= \frac{mv^2}{r} = 2\times 2240 \times \frac{(88)^2}{770} \text{ poundals} = 1408 \text{ lbs. wt.}$$

6. Here $v = \frac{176}{3}$ feet per second, and $r = 1320$ feet.

Hence, if θ be the inclination of the floor of the carriage to the horizon, we have
$$\tan\theta = \frac{v^2}{rg} = \frac{(176)^2}{9\times 1320\times 32} = \frac{11}{135}.$$

Hence the required height $= 5\sin\theta = 5\tan\theta$, nearly (since θ is small) $= \frac{11}{27}$ feet $= 4{\cdot}9$ inches.

7. Here $v = 44$ feet per second, and $r = 1200$ feet.
Hence, if θ be the inclination of the floor of the carriage to the horizon, we have
$$\tan\theta = \frac{v^2}{rg} = \frac{(44)^2}{1200\times 32} = \frac{121}{2400}.$$

Hence the required height $= 5\sin\theta = 5\tan\theta$ nearly (since θ is small) $= \frac{121}{480}$ feet $= 3{\cdot}02\ldots$ inches.

8. Here $$\tan\theta = \frac{v^2}{rg} = \frac{(66)^2}{1320\times 32} = \frac{33}{320};$$
and the required height
$$= \frac{33}{64} \text{ feet} = 6{\cdot}18\ldots \text{ inches.}$$

9. Here $v = 44$ feet per second, and $r = 300$ feet.
Hence, if θ be the inclination of the string to the vertical, we have
$$\tan\theta = \frac{v^2}{rg} = \frac{(44)^2}{300\times 32} = \frac{121}{600}.$$

Hence $$\sin\theta = \frac{121}{120\sqrt{26}} \text{ nearly,}$$
and the required distance
$$= 6\sin\theta = \frac{121}{20\sqrt{26}} \text{ feet} = \text{about 1 foot } 2\tfrac{1}{4} \text{ inches.}$$

10. In equation (3) of Art. 143, put $a=\frac{1}{2}$, and $\theta=60°$. Then we have

$$\frac{1}{2}=g\div\frac{\omega^2}{2}, \text{ so that } \omega^2=4g, \text{ i.e. } \omega=2\sqrt{g}.$$

But
$$\omega=\frac{n}{60}\times 2\pi,$$

n being the required number of revolutions; hence

$$n=\frac{60\omega}{\pi}=\frac{60}{\pi}\sqrt{g}=108, \text{ nearly.}$$

11. Let v be the velocity of the body mass m, r be its distance from the axis of the cone, and R be the normal reaction of the cone. Then we have

$$R\cos\theta=\frac{mv^2}{r}, \text{ and } R\sin\theta=mg.$$

Hence
$$\cot\theta=\frac{v^2}{rg}; \text{ but } v=n\,.\,2\pi r,$$

so that
$$r^2=(rg\cot\theta)\div 4\pi^2n^2, \text{ i.e. } r=\frac{g\cot\theta}{4\pi^2n^2}.$$

12. The velocity of the mass

$$=\frac{10}{60}\times 2\pi\times 29=\frac{29\pi}{3} \text{ feet per second;}$$

hence, at the highest point, we have

$$\frac{v^2}{r}=\frac{(29\pi)^2}{9\times 29}=g, \text{ nearly.}$$

13. Let T_1, T_2 and T_3 be the three tensions, and v be the velocity.

Then
$$v=\frac{600}{60}\times 2\pi\times 3=60\pi \text{ feet per second;}$$

and
$$T_1=mg+\frac{mv^2}{r}=m\left[32+\frac{(60\pi)^2}{3}\right],$$

$$T_2=\frac{mv^2}{r}-mg=m\left[\frac{(60\pi)^2}{3}-32\right], \text{ and } T_3=\frac{mv^2}{r}=m\,.\,\frac{(60\pi)^2}{3}.$$

Hence
$$T_1:T_2:T_3=(60\pi)^2+96:(60\pi)^2-96:(60\pi)^2$$
$$=75\pi^2+2:75\pi^2-2:75\pi^2;$$

whence
$$T_1:T_2:T_3=371:369:370.$$

14. Let O be the fixed end of the string, and P and Q be the particles at the middle point and extremity respectively. Then if v be the velocity of P, the velocity of Q is $2v$, the angular velocity of P

and Q round O being the same. Hence, if $2l$ be the length of the string, the tension in PQ

$$=\frac{m(2v)^2}{2l}=\frac{2mv^2}{l};$$

and the tension in OP

$$=\frac{mv^2}{l}+\frac{2mv^2}{l}=\frac{3mv^2}{l};$$

i.e. the tensions are as 3 : 2.

15. As in Art. 143,

$$\tan\theta=\frac{v^2}{rg},\ i.e.\ \theta=\tan^{-1}\frac{v^2}{rg}\text{ in either case.}$$

16. Let P be the pressure on the table, v be the velocity of the particle, T be the tension of the string, and a be its inclination to the vertical. Then we have

$$v=2\pi nl\sin a,$$

$$T\cos a+P=mg,\text{ and }T\sin a=\frac{mv^2}{l\sin a}.$$

Hence $$P=mg-\frac{mv^2\cos a}{l\sin^2 a}=mg-m\,.\,\frac{(2\pi nl)^2}{l}\cos a$$

$$=mg-4m\pi^2n^2b,\ \left(\text{since }\cos a=\frac{b}{l}\right),\ =m(g-4\pi^2n^2b).$$

The particle will not remain in contact with the table if P become negative; for then the table would have to pull the particle, which is impossible.

Also, P will be negative if $n>\frac{1}{2\pi}\sqrt{\frac{g}{b}}$.

17. The angular velocity ω of the umbrella

$$=\frac{14\times2\pi}{33}=\frac{14\times2}{33}\times\frac{22}{7}=\frac{8}{3}\text{ radians per second.}$$

The velocity of the drops on leaving the umbrella

$$=\frac{3}{2}\times\frac{8}{3}=4\text{ feet per second.}$$

The time that elapses before they reach the ground

$$=\sqrt{\frac{2\times4}{g}}=\frac{1}{2}\text{ second,}$$

so that the horizontal range $=4\times\frac{1}{2}=2$ feet.

Hence the required diameter $=2\sqrt{\left(\frac{3}{2}\right)^2+2^2}=5$ feet.

The normal acceleration of the drop $=\frac{3}{2}\times\left(\frac{8}{3}\right)^2=\frac{32}{3}$,

so that the resultant acceleration $=\sqrt{\left(\frac{32}{3}\right)^2+g^2}=\frac{32}{3}\sqrt{10}$

at an angle $\tan^{-1}\frac{1}{3}$ with the vertical.

Hence the required force

$$=\frac{\cdot 01}{16}\times\frac{32}{3}\sqrt{10}=\frac{\sqrt{10}}{150}=\cdot 021 \text{ of a poundal.}$$

18. The tension of the string must be equal to $2mg$, and also equal to

$$\frac{mv^2}{c};$$

hence

$$\frac{v^2}{c}=2g, \text{ so that } v=\sqrt{2gc}.$$

19. If x and y be the lengths of the portions of the strings, we have

$$\frac{mv^2}{x}=\text{the tension of the string}=\frac{m'v'^2}{y}.$$

Hence

$$x : y = mv^2 : m'v'^2.$$

20. As in Art. 140, the tension of the string

$$=4m'\pi^2n^2(c-a).$$

Since the string supports the mass m hanging freely, the tension must be $m'g$.

Hence

$$n=\frac{1}{2\pi}\sqrt{\frac{m}{m'}\cdot\frac{g}{c-a}}.$$

21. If x be the radius of the circle described, and v be the velocity, we have (equating the tensions)

$$\frac{mv^2}{x}=mg.$$

When v becomes $\frac{v}{2}$, let the lower mass become M, so that

$$\frac{mv^2}{4x}=Mg.$$

Hence $M=\frac{1}{4}m$, so that the lower mass must be diminished in the ratio 1 : 4.

22. Let the masses be m and m', v be their common velocity, T be the tension of the string, $AP=r$ and $AQ=r'$. Then we have

$$T=\frac{mv^2}{r}, \text{ and } \frac{T}{\sqrt{2}}=\frac{m'v^2}{r'\sin 45^\circ}=\frac{m'v^2}{r};$$

$$\therefore\ T=\frac{m'v^2\sqrt{2}}{r}=\frac{mv^2}{r},$$

so that $$m:m'=\sqrt{2}:1.$$

Also $$\frac{T}{\sqrt{2}}=m'g, \text{ so that } m'g=\frac{m'v^2}{r}.$$

Hence $v=\sqrt{gr}$; also $r(1+\sqrt{2})=4$, i.e. $r=4(\sqrt{2}-1)$.

The time of a revolution

$$=\frac{2\pi r}{v}=2\pi\sqrt{\frac{r}{g}}=2\pi\sqrt{\frac{4(\sqrt{2}-1)}{32}}=\frac{\pi}{2}\sqrt{2\sqrt{2}-2} \text{ seconds.}$$

23. Here $\lambda\dfrac{r-a}{a}=$ the tension of the string $=\dfrac{mv^2}{r}$.

$$\therefore\ v=\sqrt{\frac{\lambda}{m}\cdot\frac{r(r-a)}{a}}.$$

24. When the motion is steady, let the inclination of the string to the vertical be α, so that

$$T=2mg\frac{l\sec\alpha-l}{l}=2mg\frac{1-\cos\alpha}{\cos\alpha}.$$

Hence $mg=T\cos\alpha=2mg(1-\cos\alpha)$, so that $\cos\alpha=\dfrac{1}{2}$,

and $$\frac{mv^2}{l\tan\alpha}=T\sin\alpha.$$

$$\therefore\ v^2=\frac{l\sin^2\alpha}{\cos\alpha}\cdot 2g\cdot\frac{1-\cos\alpha}{\cos\alpha}=3gl, \text{ so that } v=\sqrt{3gl}.$$

25. As in Art. 148 the thrust towards the inner rail

$$=m\cos\theta\,\frac{V^2-v^2}{r},$$

where $$m=10\times 2240,\quad v=66,\quad r=1320,$$

and $$\tan\theta=\frac{33}{320}=\frac{1}{10} \text{ nearly,}$$

and hence $$\cos\theta=\frac{10}{\sqrt{101}}=\left(1+\frac{1}{100}\right)^{-\frac{1}{2}}=1-\frac{1}{200} \text{ nearly.}$$

(1) Here $V=30$ miles per hour $=44$ ft. per sec.

Therefore thrust towards inner rail

$$=10\times 2240\times\left(1-\frac{1}{200}\right)\frac{44^2-66^2}{1320}\text{ poundals}$$

$$=-\frac{10}{32}\left(1-\frac{1}{200}\right)\frac{66^2-44^2}{1320}\text{ tons wt.}=-\cdot 57\text{ tons wt.}$$

Therefore thrust is ·57 tons wt. outwards and is thus caused by the inner rail.

(2) Here $V=60$ miles per hour.

Therefore thrust towards inner rail

$$=10\times 2240\left(1-\frac{1}{200}\right)\cdot\frac{88^2-66^2}{1320}\text{ poundals}$$

$=\cdot 80$ tons wt. nearly, and it is caused by the outer rail.

EXAMPLES. XXV. (Pages 190—193.)

1. If the velocity and the tension be (1) v_1 and T_1, and (2) v_2 and T_2, then $v_1{}^2=(25)^2-2g\,.\,3=433$, so that $v_1=20\cdot 8$ feet per second, and

$$T_1=\frac{5v_1{}^2}{8}\text{ poundals}=\frac{5\times 433}{3\times 32}\text{ lbs. wt.}=22\cdot 6\text{ lbs. wt.}$$

Also $$v_2{}^2=(25)^2-2g\,.\,6=241,$$

so that $$v_2=15\cdot 5\text{ feet per second,}$$

and $$T_2+5g=\frac{5v_2{}^2}{3},\text{ so that }T_2=7\cdot 6\text{ lbs. wt.}$$

2. Let v be the required velocity, and W lbs. be the required weight; then the velocity at the highest point will be $\sqrt{v^2-2.6g}$, and the tension of the string there being zero, we have

$$5g=\frac{5(v^2-2.6g)}{3}, \text{ so that } v^2=15g=16\times 30,$$

and $v=4\sqrt{30}=21\,9$ feet per second. Also the tension at the lowest point is greatest, being equal to $5g+\frac{5v^2}{3}$; hence

$$W=\left(5+\frac{5v^2}{3\times 32}\right) \text{ lbs. wt.}=30 \text{ lbs. wt.}$$

3. If T be the required tension, and v be the required velocity, we have $v^2=2g.3$, so that $v=13{\cdot}8$ feet per second; and

$$T=mg+\frac{mv^2}{3}=mg+\frac{6mg}{3}=3mg \text{ poundals.}$$

4. (1) The velocity $=\sqrt{2g.9}=24$ feet per second

(2) The velocity $=\sqrt{2g.\frac{9}{2}}=12\sqrt{2}$ feet per second.

(3) The velocity $=\sqrt{2g.\frac{9}{3}}=8\sqrt{3}$ feet per second.

(4) The velocity $=\sqrt{2g\left(\frac{9}{2}-\frac{9}{2}\cos 60^\circ\right)}=12$ feet per second.

5. Let u be the velocity at the lowest point, so that the velocity at the highest point is $\sqrt{u^2-2g\,20}$. At the highest point the tension of the string must not become negative, so that the minimum value of u is given by

$$m.\frac{u^2-2g.20}{10}=mg,$$

m being the mass of the particle. Hence

$$u^2=50g=1600, \text{ i.e. } u=40 \text{ feet per second.}$$

Also, if T be the tension at the lowest point, then

$$m\frac{u^2}{10}=T-mg.$$

Hence $T=mg+5mg=$ six times the weight of the particle.

6. If the velocity communicated to the ball be v, and the velocity of recoil be v', then the momentum communicated

$$=36v=12\times 112v';$$

but $$v'=\sqrt{2g\times 2{\cdot}25},$$

hence $$v=\frac{112}{3}\sqrt{16\times 9}=448 \text{ feet per second.}$$

Also, if T_1 and T_2 be the tensions of either rope at the instant of discharge and when the cannon comes to rest respectively, and θ be the angle through which the ropes swing, we have

$$2T_1 = \text{wt. of } 12\text{ cwt.} + \frac{12}{g} \cdot \frac{2g \times 2{\cdot}25}{9} \text{ cwt.},$$

so that $$T_1 = \text{wt. of } 6\text{ cwt.} + \frac{6}{9} \times 4\frac{1}{2}\text{ cwt.} = \text{wt. of } 9\text{ cwt.}$$

And for T_2, the velocity of the cannon being zero, we have

$$2T_2 = \text{wt. of } (12\cos\theta)\text{ cwt.};$$

but $$9 - 9\cos\theta = 2\tfrac{1}{4}, \text{ so that } \cos\theta = \frac{3}{4}.$$

$$\therefore\ T_2 = \text{wt. of } \frac{9}{2}\text{ cwt.} = \text{wt. of } 4\tfrac{1}{2}\text{ cwt.}$$

7. Let APB be the initial position of the string, P being the particle, so that $AP=5$ feet, and $BP=29$ feet. Let Q be the position of the particle when it has described 3 feet, so that $AQ=8$ feet, and $BQ=26$ feet.

Draw PM and QN perpendicular to AB.

Then $$\frac{1}{2} \cdot 30 \cdot PM = \text{area } \triangle APB = \sqrt{32 \cdot 2 \cdot 27 \cdot 3} = 72$$

$$\frac{1}{2} \cdot 30 \cdot QN = \text{area } \triangle AQB = \sqrt{32 \cdot 2 \cdot 24 \cdot 6} = 96.$$

Hence the vertical distance described by the particle

$$= \frac{96-72}{15} = \frac{8}{5} \text{ foot.}$$

Hence the required velocity $= \sqrt{2 \cdot g \cdot \frac{8}{5}} = 10{\cdot}12$ feet per second.

8. If a particle slide inside from rest at P down the arc PA to the lowest point A, the arc being smooth and subtending an angle θ at the centre O of the circle, then the velocity v at A is given by

$$v^2 = 2g \cdot OA\,(1-\cos\theta) = 4g \cdot OA \sin^2\frac{\theta}{2}.$$

Hence v varies as $\sin\frac{\theta}{2}$. But the chord $AP = 2OA \cdot \sin\frac{\theta}{2}$, so that v varies as $\frac{AP}{2OA}$, *i.e.* varies as the chord AP.

9. If A be the highest point, P be the point where the particle leaves the circle, and AP subtend an angle θ at the centre O, and v be

the velocity at P, then the horizontal velocity at $P=v\cos\theta$, and the latus rectum of the subsequent parabolic path

$$=\frac{2}{g}\cdot v^2\cos^2\theta.$$

Now $$v^2=2g\,.\,OA\,(1-\cos\theta)=2ga\,(1-\cos\theta),$$
if $OA=a$; and at P the pressure of the circle is zero, *i.e.*

$$\frac{mv^2}{a}=mg\cos\theta, \text{ so that } v^2=ga\cos\theta;$$

thus $$ga\cos\theta=2ga\,(1-\cos\theta), \text{ whence } \cos\theta=\frac{2}{3}.$$

Therefore $$v^2=\frac{2}{3}ga,$$
and hence the latus rectum

$$=\frac{2}{g}\cdot\frac{2}{3}ga\cdot\frac{4}{9}=\frac{16}{27}a.$$

10. Let u be the velocity of impact, v and v' be the velocities of m and $2m$ directly after impact, d be the diameter of the circle, and h and h' be the heights to which m and $2m$ rise after the impact. Then

$$u^2=2gd,\quad mv+2mv'=mu,$$

i.e. $$v+v'=u, \text{ and } v-v'=-\frac{1}{3}u,$$

$$\therefore\ v'=\frac{4}{9}u, \text{ and } v=\frac{1}{9}u.$$

Hence $$2gh=v^2=\frac{u^2}{81}=\frac{2gd}{81}, \text{ so that } h=\frac{d}{81},$$

and $$2gh'=v'^2=\frac{16u^2}{81}=\frac{16}{81}\times 2gd, \text{ so that } h'=\frac{16d}{81}.$$

11. If e be the required coefficient of restitution, m be the mass of each ball, u be the velocity of the moving ball just before impact, v be its velocity just afterwards, and v' be that of the other ball, then we have

$$u^2=2g\,.\,2\,(1-\cos 60^\circ)=64, \text{ so that } u=8 \text{ feet per second;}$$

and then $$mv+mv'=mu,$$
i.e. $$v+v'=8, \text{ and } v-v'=-eu=-8e,$$
whence $$v'=4\,(1+e).$$

But $$v'^2=2g\,.\,\frac{6\frac{3}{4}}{12}=4\times 9, \text{ so that } v'=6 \text{ feet per second;}$$

hence $$4\,(1+e)=6, \text{ so that } e=\frac{1}{2}.$$

12. If the radius be r, and v be the velocity on leaving the arc, the horizontal velocity then $= v \sin 30° = \frac{v}{2}$, and the latus rectum of the parabola then described

$$= \frac{2}{g} \cdot \left(\frac{v}{2}\right)^2 = \frac{v^2}{2g};$$

but $v^2 = 2g \cdot r \sin 30°$, hence the latus rectum

$$= r \sin 30° = \frac{r}{2}.$$

13. ABC is an equilateral triangle of side l, and C describes a vertical circle of radius r

$$= l \sin 60° = \frac{l\sqrt{3}}{2}.$$

If C were projected from the lowest point P with velocity v so as just to pass the highest point Q, then at Q the tension would be zero, and the velocity $= \sqrt{v^2 - 2g \cdot 2r}$; hence

$$mg = \frac{m}{r}(v^2 - 4gr),$$

so that $v^2 = 5gr$; therefore, by hypothesis, the actual velocity v' of projection $= 2\sqrt{5gr}$. Also, if T_1 and T_2 be the greatest and least tensions in CA and CB, *i.e.* at P and Q, we have

$$2T_1 \cos 30° = \left(mg + \frac{mv'^2}{r}\right) \text{ poundals,}$$

and $$2T_2 \cos 30° = \left[-mg + \frac{m}{r}(v'^2 - 4gr)\right] \text{ poundals;}$$

hence $$T_1\sqrt{3} = \left(m + \frac{m}{gr} \cdot 4 \cdot 5gr\right) \text{ lbs. wt.,}$$

so that $$T_1 = 7m\sqrt{3} \text{ lbs. wt.;}$$

and $$T_2\sqrt{3} = \left[-m + \frac{m}{gr}(20gr - 4gr)\right] \text{ lbs. wt.,}$$

so that $$T_2 = 5m\sqrt{3} \text{ lbs. wt.}$$

Again, if when C is at R halfway between P and Q, the string BC be cut, the velocity V of C is then vertical, and the string CA $(=l)$ is horizontal, and

$$V^2 = v'^2 - 2gr = 18gr = 18g \cdot \frac{l\sqrt{3}}{2} = 2gl \cdot \frac{9\sqrt{3}}{2},$$

which is greater than $2gl$; hence C will then describe completely the vertical circle through CA with centre A, and continue to revolve round it.

14. Let $ABCDEFGH$ be the tube, AB being the lowest side horizontal, and GH the vertical side, so that the first particle is at A and the second is hanging from H. The line joining A to H is inclined at 45° to the horizontal, so that the height of H above A is

$$a \sin 45^\circ, \text{ i.e. } \frac{a}{\sqrt{2}}.$$

Hence when the first particle leaves the tube at H it is at a height $\frac{a}{\sqrt{2}}$ above its starting point A, and the other particle has then descended a distance $7a$. Hence the work done by the weights of the particles during the motion $= mg\left(7a - \frac{a}{\sqrt{2}}\right)$, where m is the mass of each particle.

There is no loss of energy during the motion since there are no impacts at the corners of the tube.

Hence, if u be the required velocity, we have, as in Art. 146,

$$\frac{1}{2} \cdot 2m \,.\, u^2 = mg\left(7a - \frac{a}{\sqrt{2}}\right),$$

and therefore $$u = \frac{1}{2}\sqrt{ga\,(28 - 2\sqrt{2})}.$$

15. The apparent weight of a body m at the equator $=$ its real weight $-\frac{mv^2}{r}$, where r is the radius of the earth in feet, and v the velocity of any point of it in feet per second, due to the earth's rotation; also, since the earth rotates once in 24 hours,

$$v=\frac{2\pi r}{24\times 60\times 60}.$$

Hence the weight is diminished by a fraction of itself

$$=\frac{4\pi^2 r}{(24)^2\times(60)^4\times g}=\frac{\pi^2\times 4000\times 1760\times 3}{(12)^2\times 36\times 360000\times 32}=\frac{1}{287}\text{, nearly.}$$

In the case of the train, its velocity in space

$$=v+\frac{1760\times 3}{60}=\frac{2\pi\times 4000\times 1760\times 3}{24\times 60\times 60}+\frac{1760\times 3}{60}=\frac{11\times 9304}{63}.$$

Hence the weight is diminished by a fraction of itself

$$=\frac{(11)^2\times(9304)^2}{(63)^2\times 32\times 4000\times 1760\times 3}=\cdot 004\text{, nearly.}$$

16. If u be the velocity at the lowest point of the path, then $u^2=2gh$. But in order that the particle may make complete revolutions we must have, as in Art. 150, $u^2>5gr$. $\therefore 2h>5r$.

17. Let the notation be as in Art. 151, so that

$$mv=mv''-mv',\ \text{and}\ v''+v'=ev.$$

Hence $$2v''=v(1+e)\ \ldots\ldots\ldots\ldots\ldots(1).$$

If l be the length of the string, we have

$$v^2=2gh=g\cdot\frac{x^2}{l},\ \text{so that}\ v=\sqrt{\frac{g}{l}}\cdot x.$$

So $v''=\sqrt{\frac{g}{l}}\cdot y$. Hence (1) gives

$$2y=x(1+e),\ \text{and}\ \therefore e=\frac{2y-x}{x}.$$

18. If v be the velocity when the string becomes tight, and θ be its inclination to the horizontal, then

$$v=\sqrt{2g\cdot 3}=8\sqrt{3},\ \text{and}\ \sin\theta=\frac{1\frac{1}{2}}{3}=\frac{1}{2},\ \text{so that}\ \theta=30^\circ.$$

The velocity along the tangent is unaltered by the tightening of the string, and $\therefore =v\cos\theta=8\sqrt{3}\times\frac{\sqrt{3}}{2}=12$ ft. per sec.

Also height to which it now rises

$$=\frac{12^2}{2g}=\frac{144}{64}=2\tfrac{1}{4}\text{ ft.}$$

$$=9\text{ inches above } O.$$

19. As in Art. 149 the angle θ required is given by making $R=0$,

and $$\therefore\ \cos\theta=\frac{v^2}{gr}$$

$$=\frac{\frac{1}{25}\times 95ga-2ga\,(1+\cos\theta)}{ga}=\frac{45}{25}-2\cos\theta,\ \text{so that}\ \cos\theta=\frac{8}{5}.$$

Also $$v^2=\frac{1}{25}\times 95ga-2ga\,(1+\cos\theta)=ga\left[\frac{95}{25}-2-\frac{6}{5}\right]=\frac{3}{5}ga,$$

so that $$v=\frac{\sqrt{15}}{5}ga.$$

20. The velocity V immediately after the impact is given by

$$40000\times\frac{1}{5}=\left(20+\frac{1}{5}\right)V,\ \text{so that}\ V=\frac{40000}{101}\ \text{cms. per sec.}$$

$$\therefore\ \text{Height required}=\frac{V^2}{2g}=\frac{16\times 10^8}{2\times 981\times 101^2}\ \text{cms.}=80\ \text{cms. nearly.}$$

21. If V be the velocity of the shot before the impact, and V' the common velocity of the box of sand and the shot immediately after, then

$$20\times V=2020\times V',\ \textit{i.e.}\ V=101\times V'.$$

Also $$V'^2=2g\,.\,h=2g\,.\,\frac{6^2}{2\times 8}=4\times 6^2,\ \text{so that}\ V'=12.$$

Hence $$V=1212\ \text{ft. per sec.}$$

EXAMPLES. XXVI. (Pages 199, 200.)

1. The time of oscillation from rest to rest $=\frac{\pi}{\sqrt{\mu}}$, where μ is the absolute acceleration.

(1) $2\mu=4$, so that $\frac{\pi}{\sqrt{\mu}}=\frac{\pi}{\sqrt{2}}=\frac{1}{2}\pi\sqrt{2}$ seconds,

(2) $\frac{3\mu}{12}=9$, so that $\mu=36$, and $\frac{\pi}{\sqrt{\mu}}=\frac{\pi}{6}$ seconds,

and (3) $1\,.\,\mu=\pi^2$, so that $\frac{\pi}{\sqrt{\mu}}=\frac{\pi}{\pi}=1$ second.

2. The velocity at a distance x from the centre $=\sqrt{\mu\,(a^2-x^2)}$, so that the velocity v at the centre $=a\sqrt{\mu}$;

(1) where $a=2$, and $\mu=2$, $v=2\sqrt{2}$ feet per second;

(2) where $a=\frac{3}{12}=\frac{1}{4}$, and $\mu=36$, $v=1\frac{1}{2}$ feet per second;

and (3) where $a=1$, and $\mu=\pi^2$, $v=\pi$ feet per second.

3. As in 2, the velocity $= a\sqrt{\mu} = \sqrt{\mu}$ here.

(1) $\frac{2\pi}{\sqrt{\mu}} = 2$, so that $\sqrt{\mu} = \pi$, and $v = \pi$ feet per second;

(2) $\frac{2\pi}{\sqrt{\mu}} = \frac{1}{16}$, so that $\sqrt{\mu} = 32\pi$, and $v = 32\pi$ feet per second;

and (3) $\frac{2\pi}{\sqrt{\mu}} = \pi$, so that $\sqrt{\mu} = 2$, and $v = 2$ feet per second.

4. We have $4 = \sqrt{\mu} . a$, and $\pi = \frac{2\pi}{\sqrt{\mu}}$,

so that $\mu = 4$ and $a = 2$. Hence, when $x = 1$, we have

$$v = \sqrt{\mu(a^2 - x^2)} = 3 \cdot 46 \text{ feet per second.}$$

5. Here $8 = \sqrt{\mu(a^2 - 9)}$, and $6 = \sqrt{\mu(a^2 - 16)}$.

Hence $\frac{a^2 - 9}{8^2} = \frac{a^2 - 16}{6^2}$, whence $a = 5$,

and by substitution, $\sqrt{\mu} = 2$; hence the period

$$= \frac{2\pi}{\sqrt{\mu}} = \pi \text{ seconds.}$$

Also the acceleration at the greatest distance from the centre

$$= \mu a = 20 \text{ ft.-sec. units.}$$

6. Here $a = \frac{1}{10}$, and $\frac{2\pi}{\sqrt{\mu}} = \frac{1}{256}$,

so that $\mu = (512\pi)^2$.

Hence the required force $= \frac{1}{10}(512\pi)^2$ dynes.

7. We have $\frac{2\pi}{\sqrt{\mu}} = 1$, so that $\mu = 4\pi^2$.

If a be the greatest amplitude, the acceleration of the shelf in its extreme position is $4\pi^2 a$. If the shelf and particle are to remain in contact, this must not be greater than the acceleration due to gravity. Hence the maximum value of a is given by $4\pi^2 a = 981$.

Hence $a = 25$ centimetres, nearly.

8. Let l feet be the length of the spring when the mass is hanging at rest; let a feet be its unstretched length and λ the modulus of elasticity, so that

$$\lambda \frac{l - a}{a} = 12 \text{ lbs. wt.} = 12g \quad \ldots \ldots\ldots\ldots\ldots\ldots (1).$$

Also, since the spring stretches 1 inch for each pound-weight of tension, we have

$$\lambda\frac{\frac{1}{12}}{a}=1.g \quad \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When the end is raised let its new position be O and let A be a point at a depth l below O.

When the mass is at a distance x below A the force acting on it

$$=\lambda\frac{(l+x)-a}{a}-12g=\lambda\frac{l+x-a}{a}-\lambda\frac{l-a}{a}, \text{ by (1)},$$

$$=\lambda\frac{x}{a}=12g.x, \text{ by (2)}.$$

Hence its acceleration $=12g.x\div 12=g.x$.

$\therefore$ time of a complete oscillation $=\frac{2\pi}{\sqrt{g}}$ seconds, by Art. 153 (3),

$$=\frac{2\pi}{\sqrt{32}}=1\cdot 11 \text{ secs. nearly}.$$

Also, since the mass was initially at rest at a depth of 4 inches below A, its amplitude is 4 inches above and below A.

9. With the notation of Art. 156, Ex. 3, the point O' is at a distance 3 inches, *i.e.* $\frac{1}{4}$ foot below A, and the mass would rest there so that

$$mg=\lambda\frac{l+\frac{1}{4}-l}{l}, \text{ and } \therefore \frac{\lambda}{l}=4m.g.$$

When the particle is at a distance x below O', the upward force

$$=\lambda\frac{(l+\frac{1}{4}+x)-l}{l}-mg=4mg\left(\tfrac{1}{4}+x\right)-mg=4mgx.$$

Hence the acceleration towards $O'=4gx$, and

$$\therefore \text{ time of a complete oscillation}=\frac{2\pi}{\sqrt{4g}}\text{ secs.}=\frac{\pi\sqrt{2}}{8}\text{ secs.}$$

10. Let the natural length of the string be a, so that the distance between the fixed points is $2a$. When the particle is displaced a distance x from the centres the stretched lengths of the two portions of the string are $a+x$ and $a-x$, and their unstretched lengths are each $\frac{a}{2}$.

Hence their tensions are

$$\lambda\frac{a+x-\frac{a}{2}}{\frac{a}{2}} \text{ and } \lambda\frac{a-x-\frac{a}{2}}{\frac{a}{2}},$$

i.e. $$\lambda\frac{a+2x}{a} \text{ and } \lambda\frac{a-2x}{a},$$

where λ is the modulus of elasticity.

$$\therefore \text{ force towards the centre} = \lambda\frac{a+2x}{a} - \lambda\frac{a-2x}{a} = \frac{4\lambda}{a}x.$$

$$\therefore \text{ acceleration towards the centre} = \frac{4\lambda}{ma}x.$$

The motion is thus simple harmonic, and the time of a complete oscillation

$$= 2\pi \div \sqrt{\frac{4\lambda}{ma}} = \pi\sqrt{\frac{ma}{\lambda}} \text{ secs.}$$

11. If A move in the straight line CNO, O being on the circle, and BN perpendicular on CO, then since B moves with uniform velocity u in the circle OB, N moves with simple harmonic motion; and since

$$BA = BC, \text{ we have } CA = x = 2CN;$$

therefore the velocity of A is double that of N, and therefore A moves also with simple harmonic motion. Also the velocity of N

$$= \omega\sqrt{a^2 - \left(\frac{x}{2}\right)^2},$$

where ω is B's angular velocity in the circle, so that $\omega = \frac{u}{a}$;

hence the velocity of A

$$= \frac{2u}{a}\sqrt{a^2 - \frac{x^2}{4}} = \frac{u}{a}\sqrt{4a^2 - x^2}.$$

EXAMPLES. XXVII. (Page 206)

1. Here $2 \cdot 5 = \pi\sqrt{\frac{l}{g}}$, l being the required length.

$$\therefore\ l = \frac{32 \cdot 2 \times 7^2 \times 6 \cdot 25}{(22)^2} = 20 \cdot 4 \text{ feet, nearly.}$$

2. Here $16 = 2\pi\sqrt{\frac{6400}{g}}$, whence $g = 987$ cm.-sec. units.

3. The time of oscillation $= \frac{671}{700} = \pi\sqrt{\frac{3}{g}}$,

whence $g = 32 \cdot 249$, approximately.

4. Given $1 = \pi\sqrt{\frac{39 \cdot 12}{12g}}$, then if l_1, l_2 and l_3 be the required lengths in feet, we have

$$(1)\ \frac{1}{2} = \pi\sqrt{\frac{l_1}{g}},\quad (2)\ \frac{1}{4} = \pi\sqrt{\frac{l_2}{g}},\ \text{and}\ (3)\ 3 = \pi\sqrt{\frac{l_3}{g}}.$$

Hence (1) $\sqrt{\frac{l_1}{g}}=\frac{1}{2}\sqrt{\frac{39\cdot12}{12g}}$,

whence $l_1=\frac{1}{4}\times 39\cdot12$ inches $=9\cdot78$ inches,

(2) $l_2=\frac{1}{4}l_1=2\cdot445$ inches,

and (3) $l_3=16l_1=4\times39\cdot12=156\cdot48$ inches.

5. Here, if x be the required number, the time of an oscillation

$$=\frac{242}{x}=\pi\sqrt{\frac{53\cdot41}{981}};$$

hence $x=11\times7\sqrt{\frac{981}{53\cdot41}}=33\sqrt{\frac{109\times7}{7\cdot63}}=33\sqrt{\frac{109}{1\cdot09}}=330.$

6. If $g=32$, the time of oscillation would be

$$=\pi\sqrt{\frac{1760\times8}{32}}=\frac{22}{7}\sqrt{3\times55}=40 \text{ seconds, nearly.}$$

7. Here $\frac{180}{183}=\pi\sqrt{\frac{37\cdot8}{12g}}$, so that

$$g=\pi^2\times\left(\frac{61}{60}\right)^2\times3\cdot15=\left(\frac{22}{7}\right)^2\times\left(\frac{61}{60}\right)^2\times\frac{63}{20}=32\cdot16, \text{ nearly.}$$

8. The time of an oscillation

$$=\pi\sqrt{\frac{4}{32}}=\frac{\pi}{2\sqrt{2}} \text{ sec.}$$

Hence the required number of oscillations

$$=(24\times60\times60)\div\frac{\pi}{2\sqrt{2}}=77756, \text{ taking } \pi=\frac{22}{7}$$

and $\sqrt{2}=1\cdot4142136.$

9. The required time

$$=2\pi\sqrt{\frac{450}{32}}=2\pi\sqrt{\frac{9\times25}{16}}=23\tfrac{4}{7} \text{ seconds.}$$

EXAMPLES. XXVIII. (Pages 211, 212.)

1. The pendulum beats in (24×3600) seconds, *i.e.* 86400 seconds, $(86400 - 20)$ times, *i.e.* 86380 times

Hence, if g_1 be the value of g at the second place, the time of oscillation there

$$=\frac{86400}{86380}\text{ second}=\pi\sqrt{\frac{l}{g_1}},$$

l being the length of the pendulum;

also $$1=\pi\sqrt{\frac{l}{g}};\text{ hence }\sqrt{\frac{g_1}{g}}=\frac{8638}{8640},$$

and $$g_1=g\left(1-\frac{2}{8640}\right)^2=32\cdot2\left(1-\frac{1}{2160}\right)\text{ approximately}=32\cdot185.$$

2. If g_1 and g_2 be the values of g at the two places, and T_1 and T_2 be the times of an oscillation, we have

$$T_1=\frac{86400}{86410}=\frac{8640}{8641},\text{ and }T_2=\frac{8640}{1639}.$$

$$\therefore\ \sqrt{\frac{l}{g_1}}=\frac{8640}{8641},\text{ and }\sqrt{\frac{l}{g_2}}=\frac{8640}{8639}.$$

$$\therefore\ \sqrt{\frac{g_1}{g_2}}=\frac{8641}{8639},\text{ so that }\frac{g_1}{g_2}=\left(1+\frac{2}{8639}\right)^2=1+\frac{4}{8639}$$

approximately, $=1\cdot00046$;

hence $g_1 : g_2=1\cdot00046 : 1.$

3. If n be the required number, we have

$$\frac{86400}{86400+n}=\pi\sqrt{\frac{l}{32\cdot25}},\text{ and }1=\pi\sqrt{\frac{l}{32\cdot09}}.$$

$$\therefore\ 1+\frac{n}{86400}=\sqrt{\frac{3225}{3209}}=\left(1+\frac{16}{3209}\right)^{\frac{1}{2}}=1+\frac{8}{3209}\text{ nearly},$$

$$\therefore\ n=\frac{86400\times 8}{3209}=215\cdot4,\text{ nearly.}$$

4. Proceeding as in Ex. 2, p. 210, we have

$$1-\left(\frac{86400}{86391}\right)^2=\pi^2\cdot\frac{x}{g},\text{ so that }x=-\frac{g}{\pi^2}\left[\left(\frac{86400}{86391}\right)^2-1\right]$$

$$= -\frac{g}{\pi^2}\left[\left(1-\frac{9}{86400}\right)^{-2}-1\right] = -\frac{32\times 7^2}{(22)^2}\left[\left(1-\frac{1}{9600}\right)^{-2}-1\right]$$

$$= -\frac{32\times 7^2}{(22)^2}\left[1+\frac{1}{4800}-1\right], \text{ approximately},$$

$$= -\frac{32\times 49}{484}\times\frac{1}{4800}\text{ feet} = -\cdot 008 \text{ inch}.$$

Hence the pendulum must be shortened by ·008 inch.

5. Here $$1-\left(\frac{86400}{86405}\right)^2 = \pi^2\cdot\frac{x}{g},$$

$$\therefore\ x = \frac{g}{\pi^2}\cdot\left[1-\left(\frac{86400}{86405}\right)^2\right] = \frac{g}{\pi^2}\cdot\left[1+\left(1+\frac{5}{86400}\right)^{-2}\right]$$

$$= \frac{g}{\pi^2}\left[1-\left(1+\frac{1}{17280}\right)^{-2}\right] = \frac{32\times 7^2}{484}\times\frac{1}{8640}\text{ feet},$$

$$= \cdot 0045 \text{ inch}.$$

6. If n and n' be the number of oscillations per day in the two cases, and T and T' be the times of oscillation, then $nT = n'T'$;

hence $$\frac{n'}{n} = \frac{T}{T'} = \pi\sqrt{\frac{l}{g}} \div \pi\sqrt{\frac{l\left(1+\frac{1}{100}\right)}{g}}$$

$$= \left(1+\frac{1}{100}\right)^{-\frac{1}{2}} = 1-\frac{1}{200}, \text{ approximately}.$$

$$\therefore\ \frac{n-n'}{n} = \frac{1}{200}, \text{ so that } n-n' = \frac{n}{200} = \frac{86400}{200} = 432.$$

7. If n be the required number, we have

$$\frac{86400}{86400-n} = \pi\sqrt{\frac{l+\frac{1}{240}}{g}}, \text{ and } 1 = \pi\sqrt{\frac{l}{g}};$$

squaring and subtracting, we have

$$\left(\frac{86400}{86400-n}\right)^2 - 1 = \frac{\pi^2}{g}\cdot\frac{1}{240}.$$

$$\therefore\ \left(\frac{86400}{86400-n}\right)^2 = 1+\frac{\pi^2}{g}\cdot\frac{1}{240}.$$

$$\therefore\ 1-\frac{n}{86400} = \left(1+\frac{\pi^2}{240g}\right)^{-\frac{1}{2}} = 1-\frac{\pi^2}{480g}, \text{ approximately};$$

hence $$n = \frac{\pi^2\times 86400}{480\times 32} = 55, \text{ nearly}.$$

8. If l be the original length of the pendulum in centimetres, we have

$$\frac{44}{21}=2\pi\sqrt{\frac{l}{g}},\ \text{and}\ \frac{33}{21}=2\pi\sqrt{\frac{l-47\cdot6875}{g}},$$

so that
$$\frac{1}{3}=\sqrt{\frac{l}{g}},\ \text{and}\ \frac{1}{4}=\sqrt{\frac{l-47\cdot6875}{g}};$$

hence
$$\frac{47\cdot6875}{g}=\frac{1}{9}-\frac{1}{16}=\frac{7}{144};$$

$$\therefore\ g=\frac{144}{7}\times47\cdot6875=144\times6\cdot8125=981.$$

9. If it lose y seconds in 24 hours, we have

$$1=\pi\sqrt{\frac{l}{g}},\ \text{and}\ \frac{86400}{86400-y}=\pi\sqrt{\frac{l\times1\cdot000233}{g}}.$$

$$\therefore\ \frac{86400-y}{86400}=\sqrt{\frac{1}{1\cdot000233}}=\left(1+\frac{233}{1000000}\right)^{-\frac{1}{2}}=1-\frac{233}{2000000}\ \text{nearly};$$

hence
$$y=\frac{233\times864}{20000}=10\ \text{seconds, nearly.}$$

10. Let g_1 be the value of g at the bottom of the mine, d be the depth of the mine, r be the radius of the earth, and y be the required number of seconds. Then we have

$$1=\pi\sqrt{\frac{l}{g}},\ \frac{86400}{86390}=\pi\sqrt{\frac{l}{g_1}},$$

and
$$g_1:g=r-d:r.$$

Hence
$$1-\frac{d}{r}=\frac{g_1}{g}=\left(\frac{8639}{8640}\right)^2=\left(1-\frac{1}{8640}\right)^2=1-\frac{1}{4320}\ \text{nearly.}$$

$$\therefore\ d=\frac{4000}{4320}\ \text{miles}=\frac{100\times1760}{108}\ \text{yards}=1630\ \text{yards, nearly.}$$

Also, if g_2 be the value of g halfway down the mine, we have

$$\frac{86400}{86400-y}=\pi\sqrt{\frac{l}{g_2}};$$

hence
$$\frac{g_2}{g}=\left(1-\frac{y}{86400}\right)^2=1-\frac{y}{43200},\ \text{nearly,}$$

and
$$\frac{g_2}{g}=\frac{r-\frac{d}{2}}{r}=1-\frac{d}{2r}\quad\text{so that } y=\frac{43200}{2\times4320}=5\ \text{seconds.}$$

11. If the acceleration of gravity at the top of the mine be g_1, and at the bottom be g_2, d be the depth of the mine, and l be the length of the simple pendulum equivalent to that of the clock, then

$$\frac{86400}{86410}=\pi\sqrt{\frac{l}{g_1}}, \text{ and } \frac{86400}{86390}=\pi\sqrt{\frac{l}{g_2}};$$

also $g_1 : g_2 = r : r-d$, r being the radius of the earth; hence

$$\frac{g_1}{g_2}=\left(\frac{8641}{8639}\right)^2=1+\frac{4}{8639} \text{ nearly}=1\cdot0005, \text{ nearly};$$

i.e.

$$g_1 : g_2 = 1\cdot0005 : 1.$$

Also
$$1\cdot0005=\frac{g_1}{g_2}=\frac{r}{r-d}=\left(1-\frac{d}{r}\right)^{-1}=1+\frac{d}{r}, \text{ nearly};$$

hence
$$d=(\cdot0005\times4000) \text{ miles}=2\cdot000 \text{ miles}.$$

[More nearly,

$$d=(\cdot000463\times4000) \text{ miles}=1\cdot852 \text{ miles}.]$$

12. If g_1 be the value of g at the top of the mountain,

$$\frac{g_1}{g}=\left(\frac{4000}{4000+\frac{1}{2}}\right)^2=\left(1+\frac{1}{8000}\right)^{-2}.$$

If n be the number of seconds lost, we have

$$\frac{86400}{86400-n}=\pi\sqrt{\frac{l}{g_1}}, \text{ and } 1=\pi\sqrt{\frac{l}{g}}.$$

Hence, by division,
$$\frac{86400-n}{86400}=\sqrt{\frac{g_1}{g}};$$

i.e.
$$1-\frac{n}{86400}=\left(1+\frac{1}{8000}\right)^{-1}=1-\frac{1}{8000} \text{ nearly}.$$

$$\therefore n=\frac{86400}{8000}=10\cdot8 \text{ seconds}.$$

If the pendulum be shortened by x feet, we have

$$1=\pi\sqrt{\frac{l-x}{g_1}}=\pi\sqrt{\frac{l}{g}}.$$

$$\therefore l-x=\frac{g_1}{\pi^2} \text{ and } l=\frac{g}{\pi^2};$$

hence
$$x=\frac{g-g_1}{\pi^2}=\frac{g}{\pi^2}\left(1-\frac{g_1}{g}\right)$$

$$=\frac{g}{\pi^2}\left[1-\left(1+\frac{1}{8000}\right)^{-2}\right]=\frac{g}{\pi^2}\left[1-1+\frac{1}{4000}\right] \text{ nearly}$$

$$=\frac{32\times49}{484}\times\frac{1}{4000} \text{ feet}=\cdot01 \text{ inch, nearly}.$$

13. If h feet be the height of the mountain, and g_1 be the value of g at its top, we have

$$g_1 : g = \frac{1}{(r+h)^2} : \frac{1}{r^2},$$

r being the radius of the earth in feet; also

$$\frac{86400}{86400-n} = \pi\sqrt{\frac{l}{g_1}}, \text{ and } 1 = \pi\sqrt{\frac{l}{g}};$$

$$\therefore \frac{86400-n}{86400} = \sqrt{\frac{g_1}{g}} = \frac{r}{r+h}.$$

$$\therefore 1 - \frac{n}{86400} = \left(1+\frac{h}{r}\right)^{-1} = 1 - \frac{h}{r}.$$

Hence $$h = \frac{nr}{86400} = \frac{n \times 4000 \times 1760 \times 3}{86400} \text{ feet}$$

$$= 245 \,.\, n \text{ feet, approximately.}$$

14. If f be the acceleration of the balloon, we have

$$900 = \frac{1}{2} \,.\, f \,.\, (60)^2, \text{ so that } f = \frac{1}{2} \text{ ft.-sec. unit.}$$

When the clock is on the ground the acceleration of the point of suspension of the pendulum relative to the earth is zero, but in the balloon it has an upward acceleration of $\frac{1}{2}$; thus in considering the relative motion of the bob, g is practically increased by $\frac{1}{2}$; hence the time of an oscillation becomes

$$\pi\sqrt{\frac{l}{32\frac{1}{2}}},$$

where $$1 = \pi\sqrt{\frac{l}{32}},$$

i.e. the time of oscillation $$= \sqrt{\frac{32}{32\frac{1}{2}}};$$

therefore the clock makes

$$\sqrt{\frac{32\frac{1}{2}}{32}} \times 3600 \text{ beats per hour instead of } 3600;$$

hence it gains

$$3600\left[-1+\left(1+\frac{1}{64}\right)^{\frac{1}{2}}\right] \text{ seconds per hour,}$$

i.e. $$3600 \times \frac{1}{128}, \text{ i.e. } 28 \text{ seconds per hour, nearly.}$$

15. As in the last example, the time of an oscillation will be,

$$\pi\sqrt{\frac{l}{32-1}},$$

the acceleration of the point of suspension being in the same direction as that of the bob; thus the time of oscillation

$$=\sqrt{\frac{32}{31}},$$

and the clock will lose per hour $3600\left(1-\sqrt{\frac{32}{31}}\right)$ seconds,

i.e. $\quad 3600\left[1-\left(1-\frac{1}{32}\right)^{\frac{1}{2}}\right]$, *i.e.* $3600\times\frac{1}{64}$,

i.e. about 56 seconds.

16. If g_1 and g be the accelerations at their surfaces due to the attractions of the moon and the earth respectively, then

$$g_1 : g=\frac{\text{mass of moon}}{(\text{its radius})^2} : \frac{\text{mass of earth}}{(\text{its radius})^2}=1 : \frac{81}{4^2}=16 : 81;$$

also $\quad 1=\pi\sqrt{\frac{l}{g}};$

hence the time of oscillation on the moon would be

$$\pi\sqrt{\frac{l}{g_1}}=1\cdot\sqrt{\frac{g}{g_1}}=\sqrt{\frac{81}{16}}=\frac{9}{4}=2\tfrac{1}{4}\text{ secs.}$$

17. The normal acceleration f of the train is $\frac{(88)^2}{x}$, where x is the radius of the curve. The tension of the string of the pendulum is therefore $\quad m\sqrt{f^2+g^2}.$

The pendulum therefore moves as if the acceleration due to gravity were g_1, where $\quad g_1=\sqrt{f^2+g^2}.$

Hence $\quad 1=\pi\sqrt{\frac{l}{g}}$, and $\frac{120}{121}=\pi\sqrt{\frac{l}{g_1}}.$

Hence $\quad \left(\frac{121}{120}\right)^4=\frac{g_1^2}{g^2}=1+\frac{f^2}{g^2},$

so that $\quad 1+\frac{f^2}{g^2}=\left(1+\frac{1}{120}\right)^4=1+\frac{1}{30}$, approximately.

$$\therefore \frac{(88)^2}{x}=f=\frac{g}{\sqrt{30}},$$

so that $\quad x=242\sqrt{30}$ feet $=$ about one quarter of a mile.

18. Let r be the radius of the earth; the attraction produces an acceleration which varies as the distance from the centre of the earth and at the surface is equal to g; therefore the particle actually has an harmonic motion, and

$$t=\frac{\pi}{2\sqrt{\mu}}, \text{ where } \mu r=g.$$

Also $$r=\frac{1}{2}gt_1^2, \text{ i.e. } t_1=\sqrt{\frac{2r}{g}};$$

hence we have $$t:t_1=\frac{\pi}{2}\sqrt{\frac{r}{g}}:\sqrt{\frac{2r}{g}}=\pi:2\sqrt{2}.$$

19. If θ be the greatest inclination of the string OP to the vertical OA, v be the velocity of the bob at A its lowest position, T be the tension of the string when at A, and $OA=r$, then we have

$$2mg=T=mg+\frac{mv^2}{r},$$

so that $$v^2=gr; \text{ but } v^2=2rg(1-\cos\theta).$$

Hence $$\theta=\frac{\pi}{3}.$$

20. Let O be the centre of the circle in which the mass swings; ABA' the arc of swing, B being the lowest point. Let AN be perpendicular to OB, so that $AN=\frac{1}{8}$ ft.

Then $$\sin BOA=\frac{AN}{OA}=\frac{1}{64},$$

so that the angle BOA is small. Hence, by Art. 159, the required time

$$=2\pi\sqrt{\frac{8}{g}}\text{ secs.}=\pi\text{ secs.}$$

The acceleration at A $$=g\sin AOB=\frac{g}{64}.$$

The velocity at B

$$=\sqrt{2g\,.\,BN}=\sqrt{2g\,.\,\frac{AB^2}{2\,.\,OB}}=\sqrt{2g\,.\,\frac{(\frac{1}{8})^2}{16}}\text{ approx.}=\tfrac{1}{4}\text{ ft. per sec.}$$

EXAMPLES. XXIX. (Page 219.)

1. We have $[L'] = 5280[L]$, and $[T'] = 60[T]$.

Hence $\dfrac{[V']}{[V]} = \dfrac{[L'][T']^{-1}}{[L][T]^{-1}} = \dfrac{5280}{60} = 88.$

Also $\dfrac{[F']}{[F]} = \dfrac{[L'][T']^{-2}}{[L][T]^{-2}} = \dfrac{5280}{(60)^2} = \dfrac{22}{15}.$

2. We have $[L'] = 5280[L]$, and $[T'] = 4[T]$.

Hence $\dfrac{[V']}{[V]} = \dfrac{[L'][T']^{-1}}{[L][T]^{-1}} = \dfrac{5280}{4} = 1320.$

Also $\dfrac{[F']}{[F]} = \dfrac{[L'][T']^{-2}}{[L][T]^{-2}} = \dfrac{5280}{4^2} = 330.$

3. Here $[V'] = 44[V]$, and $[T'] = 60[T]$.

Hence $[L'][T']^{-1} = 44[L][T]^{-1}$,

so that $[L'] = 44 \times 60[L] = 880$ yards.

Also $\dfrac{[F']}{[F]} = \dfrac{[L'][T']^{-2}}{[L][T]^{-2}} = \dfrac{44 \times 60}{(60)^2} = \dfrac{11}{15}.$

4. Here $[F'] = g[F]$, and $[T'] = 5[T]$.

Hence $[V'] = [F'][T'] = g \times 5 = 160$ feet per second.

5. We have $14[F'] = g[F]$, and $[T'] = 5[T]$.

Hence $14[L'][T']^{-2} = 32[L][T]^{-2}$.

$$\therefore\ [L'] = \frac{32}{14} \times 5^2[L] = 57\tfrac{1}{7} \text{ feet.}$$

6. Here $[V'] = \frac{22}{5}[V]$, and $[T'] = 60[T]$.

Hence
$$[L'] = [V'][T'] = \frac{22}{5} \times 60[V][T]$$
$$= 264 \text{ feet} = 88 \text{ yards.}$$

7. We have $[L'][T']^{-2} = 32[L][T]^{-2}$,

and $[L'][T']^{-1} = 160[L][T]^{-1}$.

Hence, by division, $[T'] = 5[T]$, and $[L'] = [L] \times 32 \times 5^2 = 800$ feet.

8. (1) $\frac{32}{1} \cdot \left(\frac{1}{2}\right)^2 = \frac{32}{4} = 8.$

(2) $\frac{32 \times (11)^2}{1760 \times 3} = \frac{11}{15}.$

(3) $\frac{32 \times (10 \times 60)^2}{10 \times 3} = 384000.$

9. Let x be the required measure,

so that $x[F'] = 32[F]$,

i.e. $x[L'][T']^{-2} = 32[L][T]^{-2}$.

But $[L'] = \frac{\cdot 3937}{12}[L]$, and $[T'] = 60[T]$.

$$\therefore\ x = 32 \times \frac{12}{\cdot 3937} \times 60^2 = 3511303.$$

10. Here $[T'] = \frac{3600}{10000}[T] = \frac{9}{25}[T]$,

and $[L'] = \cdot 0328[L]$.

$$\therefore\ x = 32\frac{[F]}{[F']} = 32\frac{[L][T]^{-2}}{[L'][T']^{-2}}$$
$$= 32 \times \frac{1}{\cdot 0328} \times \left(\frac{9}{25}\right)^2 = 126\tfrac{18}{41}.$$

11. We have

$$100[L']^2 = 9 \times 48400[L]^2, \text{ so that } [L'] = 66[L].$$

Also
$$58\tfrac{2}{3}[F'] = 32[F].$$

$$\therefore \frac{176}{3}[L'][T']^{-2} = 32[L][T]^{-2}.$$

Hence
$$\frac{[T']^2}{[T]^2} = \frac{176}{3 \times 32} \times 66 = (11)^2,$$

so that the unit of time is 11 seconds.

EXAMPLES. XXX. (Pages 224–226.)

1. By Art. 175 (4), we have the required unit of force

$$= \frac{112}{1} \times \frac{39}{12} \times (3)^{-2} = 40\tfrac{4}{9} \text{ poundals.}$$

2. By Art. 175 (4) and (8), we have the required unit of force

$$= 10 \times 10 \times (10)^{-2} = 1 \text{ poundal,}$$

and the required unit of work

$$= 10 \times (10)^2 \times (10)^{-2} = 10 \text{ foot-poundals.}$$

3. One pound-weight $= 32$ poundals.

The unit of length $= L' = 2L$,

the unit of force $= P' = 32P$ .... (1),

the unit of mass $= M' = M$.

From (1) we have $M'L'[T']^{-2} = 32ML[T]^{-2}$;

hence $[T']^{-2} = 16[T]^{-2}$,

and therefore $T' = \frac{1}{4}T = \frac{1}{4}$ second.

4. Since the weight of one ton produces in one cwt. an acceleration equal to $20g$, we have

$$[F'] = 20g[F],$$

i.e.
$$[L'][T']^{-2} = 20g[L][T]^{-2}.$$

Also
$$[L'] = 5280[L];$$

$$\therefore \frac{[T']^2}{[T]^2} = \frac{5280}{20g} = \frac{33}{4}.$$

Hence the new unit of time is $\frac{1}{2}\sqrt{33}$ seconds.

5. The unit of velocity $=88$ feet per second $=[L][T]^{-1}$,

the unit of acceleration $=\dfrac{88}{300}$ ft.-sec. units $=[L][T]^{-2}$,

and the unit of force $=\dfrac{32\times 2240}{2}$ poundals $=[M][L][T]^{-2}$.

Hence $$(88)^2=[L^2][T]^{-2}=[L]\frac{88}{300};$$

$$\therefore\ [L]=88\times 300 \text{ feet}=8800 \text{ yards};$$

$$[T]=\frac{88\times 300}{88}=300 \text{ seconds};$$

and $$[M]=\left(\frac{32\times 2240}{2}\div\frac{88}{300}\right) \text{lbs.}=54\tfrac{6}{11} \text{ tons.}$$

6. The unit of force $=32$ poundals $=112\times[L]\times(60)^{-2}$.

$$\therefore\ L=\frac{32\times 3600}{112} \text{ feet}=342\tfrac{6}{7} \text{ yards.}$$

7. The unit of velocity $=\dfrac{60}{9}=\dfrac{20}{3}=[L]\times(60)^{-1}$.

$$\therefore\ L=20\times\frac{60}{3}=400 \text{ feet.}$$

The unit of force in poundals

$$=32\times\frac{5}{16}=10=[M][L][60]^{-2},$$

so that $$M=\frac{10\times(60)^2}{400}=90 \text{ lbs.}$$

8. Here $[L']=\dfrac{33}{2}[L]$, $[V']=3[V]$, and $[P']=6[P]$.

The second of these gives

$$[L'][T']^{-1}=3[L][T]^{-1}, \textit{ i.e. } [T']=\frac{11}{2}[T];$$

hence, from the third, viz.

$$[M'][L'][T']^{-2}=6[M][L][T]^{-2},$$

we have $$[M']=6\times\frac{2}{33}\times\frac{(11)^2}{2^2}[M]=11 \text{ lbs.}$$

9. (i) $\dfrac{\text{One poundal}}{\text{one dyne}}=\dfrac{[M'][L'][T']^{-2}}{[M][L][T]^{-2}}=453\times 30{\cdot}5=13816{\cdot}5.$

(ii) $\dfrac{\text{One foot-poundal}}{\text{one erg}}=\dfrac{[M'][L']^2[T']^{-2}}{[M][L]^2[T]^{-2}}$

$$=453\times(30{\cdot}5)^2=421403{\cdot}25.$$

(iii) One erg $=\dfrac{1}{421403{\cdot}25}$ foot-poundals

$=\dfrac{1}{32\times 421403{\cdot}25}$ foot-pounds $=\dfrac{1}{13484904}$ foot-pounds

$=7{\cdot}416\times 10^{-8}$ foot-pounds.

(iv) $\dfrac{\text{One horse-power}}{\text{one erg per sec.}}=550\times\dfrac{\text{one foot-poundal}\times 32}{\text{one erg}}$

$$=550\times 32\times 453\times(30{\cdot}5)^2$$
$$=550\times 13484904=7{\cdot}416\times 10^9.$$

10. Here $[L][T]^{-2}=[L'][T']^{-2}$ ....(1),

and $[L'][T']^{-1}=3[L][T]^{-1}$ ....(2).

Squaring (1) and (2), and multiplying the results together, we have

$$[L][L']^2=9[L'][L]^2,$$

so that $9[L]=[L']$,

i.e. the units of length are as 1 : 9.

Also, multiplying (1) and (2) together, we have

$$[T]^{-2}[T']^{-1}=3[T']^{-2}[T]^{-1},$$

so that $3[T]=[T']$,

i.e. the units of time are as 1 : 3.

If, further, m and m' be measures of the mass of a body referred to the two units of mass, and v and v' be its velocity in the two systems, we have

$$mv:m'v'=5:2,\ \text{and}\ v'=3v,$$

so that $m=\dfrac{15}{2}m'$, and therefore $m:m'=15:2$.

Hence the units of mass are as 2 : 15. (Art. 168.)

11. We have

$$[L']=2[L],\ [V']=2[V],\ \text{and}\ [P']=2[P].$$

Hence $[T']=[L'][V']^{-1}=[L][V]^{-1}=[T]$.

Also $[M']=[P'][L']^{-1}[T']^{-2}=[P][L]^{-1}[T]^{-2}=[M]$.

And $[K]=[M'][L']^2[T']^{-2}=[M][L]^2[T]^{-2}=4[K]$.

12. We have $[T']=(60)^2[T]$, $[M']=112[M]$,

and $$[M'][L'][T']^{-2}=g[M][L][T]^{-2},$$

so that $$[L']=g\times\frac{1}{112}\times(60)^4[L]$$

Therefore $$\frac{[W']}{[W]}=\frac{[M'][L']^2[T']^{-2}}{[M][L]^2[T]^{-2}}=\frac{g^2}{112}\times(60)^4=\frac{4}{7}\times(120)^4.$$

So $$\frac{[K']}{[K]}=\frac{[M'][L'][T']^{-1}}{[M][L][T]^{-1}}=g\times(60)^2=8\times(120)^2.$$

13. The momentum $=1$. Hence

$$4\times5=1\ [M][L][T]^{-1} \qquad (1),$$

and $$\frac{1}{2}\cdot4\times5^2=1\cdot[M][L]^2[T]^{-2} \qquad (2),$$

also $$1=\frac{1}{g}\ [M][L][T]^{-2} \qquad (3).$$

From (2) and (3), $[L]=\dfrac{2\times25}{32}=1\tfrac{9}{16}$ feet,

from (1) and (3), $[T]=\dfrac{4\times5}{32}=\dfrac{5}{8}$ second;

hence, from (1), $[M]=\dfrac{20\times16\times5}{25\times8}=8$ lbs.

14. The new unit of acceleration is g times the ft.-sec. unit, the new unit of velocity $5g$ times the original, and the new unit of momentum $10g$ times the original. Hence

$$[L'][T']^{-2}=g[L][T]^{-2} \qquad (1)$$

$$[L'][T']^{-1}=5g[L][T]^{-1} \qquad (2),$$

and $$[M'][L'][T']^{-1}=10g[M][L][T]^{-1} \qquad (3).$$

From (1) and (2), by division

$$[T']=5[T],$$

and hence $$[L']=g\cdot5^2[L].$$

Therefore, from (3),

$$[M']=\frac{10}{5}[M].$$

Hence the required units are $25g$ feet, *i.e.* 800 feet, 5 seconds, and $\frac{10}{5}$ lbs., *i.e.* 2 lbs.

15. In this case

$$[W']=9\times 112\times g[W],\ [K']=16[K],$$

and

$$[F']=3g[F],$$

so that

$$[M'][L']^2[T']^{-2}=9\times 112\times g[M][L]^2[T]^{-2}\ \ldots\ \ldots (1),$$

$$[M'][L'][T']^{-1}=16[M][L][T]^{-1}\ \ldots\ldots\ \ldots (2),$$

and

$$[L'][T']^{-2}=3g[L][T]^{-2}\ldots\ldots\ \ldots (3).$$

Dividing (1) by the product of (2) and (3), we have

$$[T']=21[T]=21 \text{ seconds.}$$

Hence, from (3), $[L']=3g\times(21)^2[L]$

$$=14112 \text{ yards.}$$

Therefore, from (2),

$$[M']=16\times\frac{1}{3\times 32\times(21)^2}\times 21\,[M]=\frac{1}{126}\text{ lb.}$$

16. The new unit of acceleration is $\frac{g}{16}$ C.G.S. units.

In 4 seconds the body moves through $\frac{1}{2}\cdot\frac{g}{16}\cdot 4^2$, *i.e.* $\frac{g}{2}$ centimetres.

Hence the new unit of work is $\frac{g^2}{2}$ old units.

Also the new unit of power is $90g$ old units.

Hence

$$[L'][T']^{-2}=\frac{g}{16}[L][T]^{-2}\ldots\ldots\ \ldots\ldots (1),$$

$$[M'][L']^2[T']^{-2}=\frac{g^2}{2}[M][L]^2[T]^{-2}\ \ldots\ \ldots\ldots (2),$$

and

$$[M'][L']^2[T']^{-3}=90g[M][L]^2[T]^{-3}\ldots\ \ldots (3).$$

From (2) and (3), by division,

$$[T']=\frac{g}{180}[T]=\frac{981}{180}\text{ seconds}=5\cdot 45\text{ seconds.}$$

Hence, from (1),

$$[L']=\frac{g}{16}\cdot\frac{g^2}{(180)^2}[L]=1821\text{ centimetres, nearly.}$$

Substituting in (3), we have

$$[M']=\frac{90\times(16)^2\times 180}{g^2}[M]=\frac{90\times(16)^2\times 180}{(981)^2}[M]=4\cdot 3\text{ grammes.}$$

17. 60 miles per hour $=88$ feet per second.

We have $$8[L'][T']^{-1}=88[L][T]^{-1} \text{ .... } (1),$$
$$10[M'][L'][T']^{-2}=1600g[M][L][T]^{-2} \text{ .... } (2),$$
and $$10[M'][L']^2[T']^{-2}=1600g\times 5280[M][L]^2[T]^{-2} \text{ ...... } (3).$$

From (2) and (3),
$$[L']=5280 \text{ feet}=1 \text{ mile}.$$

Hence, from (1),
$$[T']=\frac{1}{11}\times 5280[T]=480 \text{ seconds}=8 \text{ minutes}.$$

Substituting in (3), we have
$$[M']=\frac{160\times 32\times 480}{11} \text{ lbs.}=99\tfrac{57}{77} \text{ tons}.$$

18. 3 new units of acceleration equal g old units; 100 new units of kinetic energy equal $\frac{1}{2}\times 600\times(1600)^2$ original units; and 10 new units of momentum equal 600×1600 original units. Hence
$$3[L'][T']^{-2}=g[L][T]^{-2} \text{ .... } (1),$$
$$100[M'][L']^2[T']^{-2}=\frac{1}{2}\times 600\times(1600)^2[M][L]^2[T]^{-2} \text{ ... } (2),$$
and $$10[M'][L'][T']^{-1}=600\times 1600[M][L][T]^{-1} \text{ .... } (3).$$

From (2) and (3), by division,
$$10[L'][T']^{-1}=800[L][T]^{-1} \text{ .... } (4).$$

From (4) and (1), we have
$$[T']=7\tfrac{1}{2} \text{ seconds, and } [L']=600 \text{ feet}.$$

Substituting in (3), we have
$$[M']=1200 \text{ lbs}.$$

19. 45 miles per hour $=66$ feet per second.
$$[E]=[M][L]^2[T]^{-2}=\frac{1}{11}\times\frac{1}{2}\times 2240\times 100\times(66)^2=224000\times 198,$$
$$[I]=[M][L][T]^{-1}=\frac{1}{5}\times 224000\times 66,$$
and $$[H]=[M][L]^2[T]^{-3}=\frac{40}{15}\times 550 \text{ ft. lbs.}=\frac{4400\times 32}{3} \text{ ft.-poundals}.$$

Hence $$[T]=\frac{224000\times 198}{\left(\frac{4400\times 32}{3}\right)}=\frac{280\times 3\times 18}{16} \text{ seconds},$$

i.e. the unit of time $=\dfrac{280\times 3\times 18}{60\times 16}$ minutes $=15\frac{3}{4}$ minutes.

Also $$[L][T]^{-1}=\frac{1}{22}\times 66\div\frac{1}{5}=15;$$

$$\therefore\ [L]=15[T]=\frac{15\times 280\times 3\times 18}{16}\text{ feet,}$$

i.e. the unit of length $=\dfrac{15\times 280\times 3\times 18}{16\times 3}$ yards

$$=\frac{4725}{1760}\text{ miles}=2\tfrac{241}{352}\text{ miles.}$$

Also $$[M][L][T]^{-1}=15[M]=\frac{224000\times 66}{5}\text{ lbs.,}$$

i.e. the unit of mass $=\dfrac{6600}{5\times 15}=88$ tons.

Also $g=32$ ft.-sec. units

$$=32\times\frac{16}{15\times 280\times 3\times 18}\times\frac{(63)^2\times(60)^2}{4^2}=2016.$$

20. Since the units of length and time are unaltered, the units of force and mass must be altered proportionately. But the new unit of force is g poundals; hence the new unit of mass is g pounds.

MISCELLANEOUS EXAMPLES. (Pages 230–240.)

1. Let t secs. be the whole time of falling, so that in $(t-1)$ secs. the particle fell $\frac{4}{9}$ of the distance which it fell in t secs. Hence

$$\frac{1}{2}g(t-1)^2=\frac{4}{9}\times\frac{1}{2}gt^2.$$

$$\therefore\ t-1=\pm\frac{2}{3}t.$$

$$\therefore\ t=3,\text{ or }\frac{3}{5}.$$

The latter value is inadmissible, and the required height

$$=\frac{1}{2}gt^2=\frac{1}{2}\times 32\times 3^2=144\text{ feet.}$$

2. Let t secs. be the time the first stone took to drop, so that the first height is $\frac{1}{2}gt^2$. The time the second stone took is then $\left(t+\frac{1}{2}\right)$ secs., and the height therefore is $\frac{1}{2}g\left(t+\frac{1}{2}\right)^2$.

$$\therefore \frac{1}{2}g\left(t+\frac{1}{2}\right)^2 = 100 + \frac{1}{2}gt^2.$$

$$\therefore \frac{1}{2}g\left(t+\frac{1}{4}\right) = 100.$$

$$\therefore t = \frac{100}{16} - \frac{1}{4} = 6 \text{ secs.}$$

Also the required height $=\frac{1}{2}g \,.\, t^2 = 576$ feet.

3. Let f be the retardation in ft.-sec units, so that

$$600^2 = 1200^2 - 2 \,.\, f \,.\, \frac{1}{12}.$$

$$\therefore f = 6\,(1200^2 - 600^2) = 18 \times 600^2.$$

If x ft. be the required distance, then

$$0 = 1200^2 - 2f \,.\, x.$$

$$\therefore x = \frac{1200^2}{2f} = \frac{1200^2}{36 \times 600^2} = \frac{4}{36} = \frac{1}{9} \text{ ft.} = 1\tfrac{1}{3} \text{ inch.}$$

4. The total masses on the two scale-pans are respectively 12 and 15 ozs.

Hence, by Art. 74, the acceleration $= \frac{15-12}{15+12}g = \frac{g}{9}$.

Let P_1 be the pressure on the 5 oz. mass, so that the total upward force on it is $P_1 - \frac{5}{16}g$ poundals. Hence

$$P_1 - \frac{5g}{16} = mf = \frac{5}{16} \times \frac{g}{9}.$$

$$\therefore P_1 = \frac{10}{9} \times \frac{5}{16}g = \frac{50}{9} \text{ oz. wt.} = 5\tfrac{5}{9} \text{ oz. wt.}$$

So, if P_2 be the pressure on the 8 oz. mass, the total downward force on it is $\frac{8}{16}g - P_2$ poundals, and

$$\therefore \frac{8}{16}g - P_2 = \frac{8}{16} \times \frac{g}{9},$$

so that $$P_2 = \frac{8}{9} \times \frac{8}{16}g \text{ poundals} = \frac{64}{9} \text{ oz. wt.} = 7\tfrac{1}{9} \text{ oz. wt.}$$

5. Let m be either of the original masses, so that the tension of the string was mg poundals. After the alterations, the masses are

$$\left(1+\frac{1}{n}\right)m \text{ and } \left(1-\frac{1}{n+2}\right)m, \text{ i.e. } \frac{n+1}{n}m \text{ and } \frac{n+1}{n+2}m.$$

Hence, by Art. 74, the tension

$$=\frac{2m_1m_2}{m_1+m_2}g=\frac{2\times\frac{n+1}{n}\times\frac{n+1}{n+2}}{\frac{n+1}{n}+\frac{n+1}{n+2}}mg$$

$$=\frac{2(n+1)}{n+2+n}mg=mg. \quad \therefore \text{ etc.}$$

6. As long as the mass $3m$ is on the table the acceleration

$$=\frac{m}{m+3m}g=\frac{g}{4} \text{ [Art. 75].}$$

Hence the common velocity when it leaves the table

$$=\sqrt{2fs}=\sqrt{2\times\frac{g}{4}\times a}.$$

Also the time t that elapses before it hits the floor is given by

$$a=\frac{1}{2}gt^2, \text{ so that } t=\sqrt{\frac{2a}{g}}.$$

During this time the horizontal velocity of $3m$ is unaltered, and hence the required distance

$$=t\times\sqrt{2\times\frac{g}{4}\times a}=a.$$

7. When the particle has finished describing the first 100 feet, let it have been falling t secs. from rest, so that

$$\frac{1}{2}gt^2-\frac{1}{2}g(t-1)^2=100,$$

and $$\therefore t=\frac{29}{8}.$$

Let x be the time taken for the next 100 feet.

Hence $$\frac{1}{2}g(t+x)^2-\frac{1}{2}gt^2=100.$$

$$\therefore (t+x)^2=t^2+\frac{25}{4}=\frac{1241}{64}.$$

$$\therefore x+\frac{29}{8}=\frac{\sqrt{1241}}{8}=\frac{35\cdot22\ldots}{8}, \text{ i.e. } x=\frac{6\cdot22\ldots}{8}=\cdot77\ldots \text{ secs.}$$

In the second case, let f be the resistance of the air. At the commencement of the second 100 feet the velocity of the particle

$$=gt=32\times\frac{29}{8}=116.$$

Hence
$$100=116\times\cdot 9+\frac{1}{2}(g-f)\times(\cdot 9)^2.$$

$$\therefore\ (f-g)\times\frac{\cdot 81}{2}=104\cdot 4-100=4\cdot 4.$$

$$\therefore\ f=g+\frac{8\cdot 8}{\cdot 81}.$$

$$\therefore\ \frac{f}{g}=1+\frac{8\cdot 8}{\cdot 81\times 32}=1+\frac{55}{162}=\frac{217}{162}.$$

8. As in the figure of page 168 let P be the position of the bob at any time, and v its velocity then, so that, by Art. 146,

$$v^2=2g\,.\,ON=2g\,.\,l\cos\theta,$$

where l is the length of the string. If T be its tension, then, by Art. 135,

$$T-mg\cos\theta=\frac{mv^2}{l}=2mg\cos\theta.$$

$$\therefore\ T=3mg\cos\theta=\frac{3mg}{l}\times ON.\quad\therefore\ \text{etc.}$$

9. When the particle is at the end of the horizontal diameter its velocity $v=\sqrt{6gr-2gr}=2\sqrt{gr}$, and the tension T_1 then

$$=m\frac{v^2}{r}=4mg.$$

When the particle is at its highest point, the velocity v_1

$$=\sqrt{6gr-2g\,.\,2r}=\sqrt{2gr},$$

and the tension T_2 is given by

$$T_2+mg=m\frac{{v_1}^2}{r}=2mg,$$

so that $T_2=mg$. Hence $T_1=4T_2$.

10. Let P be the pull of the engine in poundals, so that the acceleration of the train is $\frac{P}{m}-g\sin a-\mu g\cos a$.

$$\therefore\ v=\left(\frac{P}{m}-g\sin a-\mu g\cos a\right)t,$$

i.e.
$$P=m\left[\frac{v}{t}+g(\sin a+\mu\cos a)\right].$$

Also the space moved through in time t = product of the time and the average velocity $=\frac{1}{2}vt$.

Hence if x be the required average H.-P., then

$$x \times 550g \times t = \text{work done} = \frac{1}{2}vt \times P.$$

$$\therefore\ x = \frac{v}{1100}\frac{P}{g} = \text{etc.}$$

11. If f be the absolute acceleration of the lift upwards, and u_1 its velocity when the particle is thrown, then the total vertical distance described in time t by the particle and lift is the same.

$$\therefore\ u_1t + \frac{1}{2}ft^2 = (u+u_1)\,t - \frac{1}{2}gt^2.$$

$$\therefore\ f = \frac{2u}{t} - g = \frac{2u - gt}{t}.$$

12. Draw a line NOS from north to south, making $NO = OS$, so that ON and OS represent the velocity u of the two ships. Draw OK in a direction between north and east to represent the direction and magnitude of the velocity of the wind, and complete the parallelograms $OKMN$ and $OKLS$. Then OL and OM represent the relative velocities of the wind with respect to the two ships.

We are given

$$\angle SOL = 67\tfrac{1}{2}°, \text{ and } \angle NOM = 22\tfrac{1}{2}°.$$

$$\therefore\ \angle LOM = 90°.$$

Since $KL = KM$, and $\angle LOM$ is a right angle,

$$\therefore\ OK = KM = ON,$$

i.e. the velocity of the wind and ship are the same.

Also since $KO = KM$,

$$\therefore\ \angle KOM = \angle KMO = \angle NOM = 22\tfrac{1}{2}°.$$

$\therefore\ \angle NOK$ is 45°, and OK is drawn in the direction N.E.

13. 15 miles per hour = 22 ft. per sec. The friction must be at least capable of producing the required normal acceleration, so that if μ be the least value of the coefficient of friction, then

$$\mu mg = m\frac{22^2}{60}.$$

$$\therefore\ \mu = \frac{22 \times 22}{60 \times 32} = \frac{121}{480} = \text{about } \frac{1}{4}.$$

14. If u be the required velocity, which is destroyed in $60n$ secs., the retardation $= \dfrac{u}{60n}$.

Also from the formula $v^2=u^2+2fs$, the retardation $=\frac{u^2}{2s}$. Equating these two values, we have $u=\frac{s}{30n}$.

15. Since the straight line joining the two ships is always east and west, their velocities resolved north and south are equal. Hence since their total velocities are equal, the inclinations of their directions to the south must be the same. Hence the second vessel must be going either S.E. or S.W. Also since the vessels are approaching one another the required direction must be S.W.

16. The velocity parallel to the plane is unaltered by the impacts, so that the distance described parallel to the plane will be zero at the end of a time t given by

$$0=v\cos(\theta-\alpha)\,t-\frac{1}{2}g\sin\alpha t^2,$$

so that

$$t=\frac{2v}{g}\frac{\cos(\theta-\alpha)}{\sin\alpha}.$$

Also, since the elasticity is perfect, the velocity perpendicular to the plane is just reversed at each impact. The time of flight for each trajectory is thus twice the time in which the velocity $v\sin(\theta-\alpha)$ is destroyed by $g\cos\alpha$, and thus $=\frac{2v\sin(\theta-\alpha)}{g\cos\alpha}$.

Clearly the particle will return to the point of projection if the first of these is some multiple, n, of the second, *i.e.* if

$$\frac{2v}{g}\frac{\cos(\theta-\alpha)}{\sin\alpha}=n\times\frac{2v}{g}\frac{\sin(\theta-\alpha)}{\cos\alpha},$$

i.e. if $\cot\alpha\,.\cot(\theta-\alpha)$ is an integer.

17. The velocities of the particle at the middles of these successive intervals t are respectively

$$\frac{1}{2}ft,\quad \left(f-\frac{1}{2}f'\right)t,\quad \left(\frac{3}{2}f-f'\right)t,\quad \left(2f-\frac{3}{2}f'\right)t, \ldots.$$

Hence the total space described

$=t\times$ sum of these velocities

$$=t^2\left[\frac{1}{2}f+\left(f-\frac{1}{2}f'\right)+\left(\frac{3}{2}f-f'\right)+\left(2f-\frac{3}{2}f'\right)+\ldots\right]$$

$=t^2\times$ sum of an A.P. whose 1st term is $\frac{1}{2}f$ and whose common difference is $\frac{1}{2}(f-f')$

$$=t^2\times\frac{2n}{2}\left[f+(2n-1)\times\frac{1}{2}(f-f')\right]$$

$$=\frac{nt^2}{2}[(2n+1)f-(2n-1)f'].$$

18. The friction μmg must be at least capable of communicating the required normal acceleration $\omega^2 a$ to the particle. Hence the greatest value of ω is given by $m\omega^2 a = \mu mg$. Hence the greatest number of revolutions per minute

$$=60 \times \frac{\omega}{2\pi} = \frac{30}{\pi}\sqrt{\frac{\mu g}{a}}.$$

19. As in Art. 107 let a and $90° - a$ be the two directions of projection corresponding to a velocity of projection u, so that

$$\frac{u^2 \sin 2a}{g} = R.$$

Also $$h = \frac{u^2 \sin^2 a}{2g}, \quad \text{and} \quad h' = \frac{u^2 \sin^2 (90° - a)}{2g} = \frac{u^2 \cos^2 a}{2g}.$$

Thus $$\sqrt{hh'} = \frac{u^2}{2g} \sin a \cos a = \frac{1}{4}\frac{u^2}{g} \sin 2a = \frac{1}{4} R.$$

20. Let u be the velocity of the person as he passes through the lowest point of his swing. The tension then $= mg + m\frac{u^2}{l}$, where l is the length of the rope. If this tension be $2mg$, we have

$$2mg = mg + m\frac{u^2}{l},$$

so that $u^2 = lg$. With the figure of page 188, the velocity will then be zero at P, where $0 = u^2 - 2g \,.\, AN$, so that $AN = \frac{l}{2}$, and thus $ON = \frac{l}{2}$.

Hence $$\angle AOP = \cos^{-1}\frac{ON}{OP} = \cos^{-1}\frac{1}{2} = 60°.$$

21. If x be the distance of m' from the ring, the tension of the string is $m'\omega^2 x$ (Art. 135, Cor.). Since m remains at rest this tension must be equal to its weight mg. $\therefore m'\omega^2 x = mg$.

22. Let P be the impulse of the blow, u the velocity of each ball after impact along the line of centres and v the velocity of the first ball perpendicular to the line of centres. Then

$$P \cos a = (m + m')u \quad \text{and} \quad P \sin a = mv.$$

Hence the kinetic energy generated

$$= \frac{1}{2}(m + m')u^2 + \frac{1}{2}mv^2 = \frac{1}{2}\frac{P^2 \cos^2 a}{m + m'} + \frac{1}{2}\frac{P^2 \sin^2 a}{m}$$

$$= \frac{P^2}{2m(m + m')}[m \cos^2 a + (m + m')\sin^2 a]$$

$$= \frac{P^2}{2(m + m')}\left[\frac{m + m' \sin^2 a}{m}\right].$$

If the balls had been interchanged, the kinetic energy would similarly have been

$$\frac{P^2}{2(m+m')}\left[\frac{m'+m\sin^2\alpha}{m'}\right]. \quad \therefore \text{ etc.}$$

23. Let α be the angle the direction of projection makes with the horizontal. As in Art. 109, the time of flight

$$=\frac{2u}{g}\cdot\frac{\sin(\alpha-\beta)}{\cos\beta}=T.$$

The velocity parallel to the plane when the particle strikes it

$$=u\cos(\alpha-\beta)-g\sin\beta\,.\,T$$

$$=u\left[\cos(\alpha-\beta)-2\,\frac{\sin\overline{\alpha-\beta}\sin\beta}{\cos\beta}\right].$$

Also the velocity perpendicular to the plane then is $u\sin(\alpha-\beta)$, and by the question these two component velocities are equal. Equating them, we have

$$1=\tan(\alpha-\beta)[1+2\tan\beta], \quad i.e. \quad \frac{1}{1+2\tan\beta}=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

$$\therefore \tan\alpha=\frac{1+\tan\beta+2\tan^2\beta}{1+\tan\beta}.$$

The vertical height required

$$=PQ\sin\beta=\frac{2u^2\sin\beta}{g\cos^2\beta}\cos\alpha\sin(\alpha-\beta)=\frac{2u^2}{g}\tan\beta\,.\,\frac{\tan\alpha-\tan\beta}{1+\tan^2\alpha}$$

$$=\frac{2u^2}{g}\tan\beta\,(1+\tan\beta)\,\frac{1+\tan\beta+2\tan^2\beta-\tan\beta\,(1+\tan\beta)}{(1+\tan\beta)^2+(1+\tan\beta+2\tan^2\beta)^2}$$

$$=\frac{2u^2}{g}\tan\beta\,(1+\tan\beta)\,\frac{1+\tan^2\beta}{[2\tan^2\beta+2\tan\beta+1]\,[2+2\tan^2\beta]}$$

$$=\frac{u^2}{g}\,\frac{\tan\beta\,(1+\tan\beta)}{1+2\tan\beta+2\tan^2\beta}=\text{etc.}$$

24. Let θ be the inclination of the direction of projection to the horizontal, and x the distance from the wall of the point of projection, so that the time to the wall $=\frac{x}{V\cos\theta}$. After the impact the horizontal velocity is $eV\cos\theta$, so that the particle will be vertically over the starting point again in time $\frac{x}{eV\cos\theta}$ from the impact.

The vertical velocity is unaltered by the impact, so that the particle will be on the same vertical level as the point of projection in time $\frac{2V\sin\theta}{g}$.

The particle will therefore return to the point of projection if

$$\frac{2V\sin\theta}{g}=\frac{x}{V\cos\theta}+\frac{x}{eV\cos\theta},$$

i.e. if $$2\sin\theta\cos\theta\,.\,\frac{V^2}{g}=x\frac{1+e}{e},$$

i.e. if $$x=\frac{e}{1+e}\frac{V^2}{g}\sin 2\theta.$$

Now the greatest value of $\sin 2\theta$ is unity, so that the greatest value of x is $\frac{e}{1+e}\frac{V^2}{g}$.

25. Immediately after the impact let u be the velocity of each particle along the direction of the string.

Then the impulsive tension T of the string has changed the velocity $v'\cos\alpha$ to u, and hence

$$T=m(u-v'\cos\alpha)\text{..........................(1)}.$$

Similarly it has decreased the velocity $v\cos\alpha$ to u, and

$$\therefore\ T=m'(v\cos\alpha-u)\text{..........................(2)}.$$

Eliminating u from (1) and (2), by multiplying them by m' and m and adding, we have

$$(m'+m)\,T=mm'(v-v')\cos\alpha.$$

26. With the notation and results of Ex. 31, page 97 of the text, the acceleration of the particle down the plane face is $g\sin\alpha$, and that of the plane, resolved in a direction parallel to the face and upwards, is $f_2\cos\alpha$.

Hence the acceleration of the particle relative to the face

$$=g\sin\alpha+f_2\cos\alpha$$

$$=g\sin\alpha+\frac{mg\sin\alpha\cos^2\alpha}{M+m\sin^2\alpha}=\frac{M+m}{M+m\sin^2\alpha}\cdot g\cos\alpha.$$

27. Let f_1 be the acceleration of the wedge horizontally, and f_2 and f_3 the accelerations of the particle perpendicular to and along the face downwards.

Since the particle has no vertical acceleration,

$$\therefore\ f_3\sin\alpha+f_2\cos\alpha=0\text{..........................(1)}.$$

Also $$f_3=g\sin\alpha\text{..........................(2)}.$$

Since the plane and particle remain in contact, their accelerations perpendicular to the face are the same.

$$\therefore\ f_1\sin\alpha=-f_2\text{..........................(3)}.$$

Solving (1), (2), (3), we have

$$f_1=g\tan\alpha,\ \text{ and }\ f_2=-g\sin\alpha\tan\alpha.$$

Also, if R be the required pressure, we have

$$mg\cos a - R = mf_2 = -mg\sin a\tan a.$$

$$\therefore\ R = mg[\cos a + \sin a\tan a] = mg\sec a.$$

28. Let O be the fixed end of the string, and A and B the positions of the masses m_2 and m_1 so that $OA = AB = l$. Let T and T_1 be the tensions of the portions OA and AB, and ω the angular velocity of the string. Then

$$T - T_1 = m_2\omega^2 l, \text{ and } T_1 = m_1\omega^2 . 2l.$$

$$\therefore\ T = \omega^2 l[2m_1 + m_2]. \quad \therefore \text{ etc.}$$

29. Let A, B, C be the masses of 1, 2, and 3 lbs. and let f_1, f_2, f_3 be their accelerations, *all measured positively downwards*. Let T and T_1 be the tensions of the strings which pass round the fixed and movable pulleys D and E. Then we have

$$3g - T = 3f_3 \quad\ldots\ldots(1),$$

$$2g - T_1 = 2f_2 \quad\ldots\ldots(2),$$

$$g - T_1 = f_1 \quad\ldots\ldots(3).$$

Also the forces acting on the pulley E must balance; for otherwise, since it is of zero mass, its acceleration would be infinite, which is clearly impossible.

Hence $$T - 2T_1 = 0 \quad\ldots\ldots(4).$$

Also since the length of the string AEB is constant,

$\therefore$ depth of A + depth of B − 2 depth of E [all below D] is constant.

$\therefore$ velocity of A + velocity of B − 2 velocity of E is zero.

$\therefore$ acceleration of A + acceleration of B − 2 acceleration of E is zero,

i.e. $$f_1 + f_2 + 2f_3 = 0 \quad\ldots\ldots(5),$$

[for the acceleration of E downwards = that of C upwards = $-f_3$].

We want the values of $2T$ and f_3.

(2) and (3) give $$2f_2 - f_1 = g \quad\ldots\ldots(6).$$

(1), (2), (3), and (4) give

$$f_1 + 2f_2 - 3f_3 = 0 \quad\ldots\ldots(7).$$

Solving (5), (6), (7), we have $f_3 = \frac{g}{17}$, so that the acceleration of the greater mass is $\frac{g}{17}$ downwards.

Substituting in (1), we have

$$T = 3g - \frac{3g}{17} = \frac{48}{17}g.$$

$\therefore$ pressure on fixed pulley

$$= 2T = \frac{96}{17}g = 5\tfrac{11}{17}g = 5\tfrac{11}{17} \text{ lbs. wt.}$$

30. Let f be the acceleration of the lower mass downwards.

By the principle of work the upper mass moves 8 times as fast as the lower, and hence its acceleration is $8f$ upwards.

Let T be the tension of the string round the lower pulley; then $\frac{T}{4}$ is the tension of the string attached to the upper mass. Hence

$$28g - 2T = 28f,$$

$$\text{and } 3g - \frac{T}{4} = 3 \times (-8f) = -24f.$$

Hence, eliminating T, we have

$$28g - 24g = 28f + 192f = 220f, \quad \text{i.e. } f = \frac{g}{55}.$$

31. At a time t from the initial instant, the distances of the trains from the crossing are $a - ut$ and $b - vt$. Hence the distance x between them then is given by

$$\begin{aligned} x^2 &= (a-ut)^2 + (b-vt)^2 - 2(a-ut)(b-vt)\cos a \\ &= t^2[u^2+v^2-2uv\cos a] - 2t[au+bv-(av+bu)\cos a] \\ &\qquad\qquad + a^2+b^2-2ab\cos a \\ &= At^2 - 2Bt + C, \text{ for brevity,} \\ &= \frac{1}{A}[At-B]^2 + C - \frac{B^2}{A}. \end{aligned}$$

Now the square on the right-hand side cannot be negative and hence its least value is zero.

Hence the least value of x^2 is $C - \frac{B^2}{A}$,

i.e. $$\frac{[u^2+v^2-2uv\cos a][a^2+b^2-2ab\cos a] - [au+bv-(av+bu)\cos a]^2}{u^2+v^2-2uv\cos a},$$

i.e. $$\frac{(av-bu)^2\sin^2 a}{u^2+v^2-2uv\cos a}, \text{ on reduction.}$$

32. At the end of time t the distances between the two points measured along and perpendicular to the line a are $a - ut$ and vt.

Hence, if x be the distance between them,

$$\begin{aligned} x^2 &= (a-ut)^2 + v^2t^2 = a^2 - 2aut + V^2t^2 \\ &= V^2\left[t - \frac{au}{V^2}\right]^2 + a^2 - a^2\frac{u^2}{V^2} \\ &= V^2\left[t - \frac{au}{V^2}\right]^2 + \frac{a^2v^2}{V^2}. \end{aligned}$$

Since the smallest value of a square is zero, the least value of x^2 is when $t = \frac{au}{V^2}$ and then the value of x^2 is $\frac{a^2v^2}{V^2}$.

33. Let f_1 be the acceleration of the hanging mass M, and hence that of the pulley; let f_2 and f_3 be the accelerations of the masses M and $M+m$ on the table.

Let T be the tension of the hanging string, so that, since the pulley is massless, the tension of the other string is $\frac{1}{2}T$.

Hence $$Mg - T = Mf_1 \quad \ldots\ldots (1),$$

$$\frac{1}{2}T = Mf_2 \quad \ldots\ldots (2),$$

and $$\frac{1}{2}T = (M+m)f_3 \quad \ldots\ldots (3).$$

Also, since the length of the string on the table is constant, we have, as in Ex. 29,

$$f_2 + f_3 - 2f_1 = 0 \quad \ldots\ldots (4).$$

Substituting from (1), (2), (3), in (4) we have

$$\frac{T}{2M} + \frac{T}{2(M+m)} = 2g - \frac{2T}{M}.$$

$$\therefore\ T = \frac{4M(M+m)g}{6M+5m}.$$

$$\therefore\ f_1 = g - \frac{T}{M} = g - \frac{4(M+m)}{6M+5m}g = \frac{2M+m}{6M+5m}g.$$

34. Initially the weight and power were masses of $7m$ and m respectively, so that by the principle of work the latter would move through 7 times the distance that the former did. When m is changed to $4m$ this same relation must hold between their accelerations. Let then f and $7f$ be the accelerations of $7m$ and $4m$. Let T be the tension of the string attached to $4m$, so that $2T$ and $4T$ are the tensions of the other strings. We then have

$$4mg - T = 4m \cdot 7f = 28mf,$$

and $$7T - 7mg = 7mf.$$

Solving, we have $$f = \frac{3g}{29}.$$

35. Let f be the acceleration of the 3 lb. downwards, so that $2f$ is the acceleration of the 1 lb. up; also let T be the tension of the string.

Then $$3g - 2T = 3f,$$

and $$T - 1 \cdot g = 1 \cdot 2f.$$

Solving, we have $f = \frac{g}{7}$, and $T = \frac{9}{7}g = 1\frac{2}{7}$ lbs. wt.

36. Let A be the cyclist, B the pedestrian; when the latter hears the bell let him move so that he meets the cyclist's path in C. Draw BN $(=d)$ perpendicular to AC. Let $AB=x$, $AC=y$.

Then $$BC=x^2+y^2-2y\sqrt{x^2-d^2}.$$

The cyclist will then just meet the pedestrian if $\dfrac{y}{V}=\dfrac{BC}{v}$,

i.e. if $$v^2y^2=V^2[x^2+y^2-2y\sqrt{x^2-d^2}].$$

$$\therefore\ (V^2-v^2)\,y^2-2yV^2\sqrt{x^2-d^2}+V^2x^2=0.$$

This has unreal roots, *i.e.* the collision is impossible, if

$$[2V^2\sqrt{(x^2-d^2)}]^2<4\,(V^2-v^2)\times V^2x^2,$$

i.e. if $$4V^2v^2x^2-4V^4d^2<0,$$

i.e. if $$x<\frac{Vd}{v}.$$

37. Let the second stone be thrown at a time T after the first so as to hit the latter at a time t after it started. Then the horizontal and vertical distances described by the first in time t must be equal to the corresponding distances described by the second in time $t-T$.

$$\therefore\ V\cos\alpha\,.\,t=V'\cos\alpha'\,(t-T)\qquad\qquad(1),$$

and $$V\sin\alpha\,.\,t-\frac{1}{2}gt^2=V'\sin\alpha'\,(t-T)-\frac{1}{2}g\,(t-T)^2\qquad\qquad(2).$$

Dividing (2) by $t-T$, and substituting from (1), we have

$$V'\cos\alpha'\tan\alpha-\frac{1}{2}gt\,\frac{V'\cos\alpha'}{V\cos\alpha}=V'\sin\alpha'-\frac{1}{2}g\,(t-T).$$

$$\therefore\ V'\frac{\sin(\alpha-\alpha')}{\cos\alpha}-\frac{1}{2}gT=\frac{1}{2}gt\,\frac{V'\cos\alpha'-V\cos\alpha}{V\cos\alpha}=\frac{1}{2}g\,\frac{V'\cos\alpha'\,T}{V\cos\alpha}.$$

$$\therefore\ T=\frac{2}{g}\,\frac{VV'\sin(\alpha-\alpha')}{V\cos\alpha+V'\cos\alpha'}.$$

38. Let F be the resistance, so that in the first case we have

$$mV^2=2Ft\qquad\qquad(1).$$

In the second case, let U be the common velocity of the shot and plate when the penetration is completed, and let x be the distance penetrated. Then

$$mV^2-mU^2=2Fx\qquad\qquad(2),$$

and $$MU^2=2Fx\qquad\qquad(3).$$

From (2) and (3),

$$MmV^2=2Fx\,(M+m)=\frac{2x}{t}\,mV^2\,(M+m)\ \text{ by (1)},$$

$$\therefore\ x=t\,\frac{M}{M+m}.$$

39. Let f_1, f_2, f_3 be the accelerations, all measured downwards, of the three masses P, Q, R and let T be the tension of the string. Then

$$Pg - T = Pf_1 \text{ .... } (1),$$

$$Qg - T = Qf_2 \text{ .... } (2),$$

and

$$Rg - 2T = Rf_3 \text{ .... } (3).$$

Since the total length of the string is constant, therefore, as in Ex. 29,

$$f_1 + f_2 + 2f_3 = 0 \text{ .... } (4).$$

Substituting in (4) from (1), (2), and (3) we have

$$g - \frac{T}{P} + g - \frac{T}{Q} + 2g - \frac{4T}{R} = 0.$$

$$\therefore\ T = \frac{4PQR}{QR + RP + 4PQ} g.$$

Also from (3)

$$f_3 = g - \frac{2T}{R} = g - \frac{8PQ}{QR + RP + 4PQ} g = \frac{QR + RP - 4PQ}{QR + RP + 4PQ} g.$$

40. As in Ex. 26 the acceleration f of the particle relative to the plane face is $\frac{M+m}{M + m\sin^2 a} \cdot g \sin a$.

Also the particle goes a distance $\frac{h}{\sin a}$ along the plane;

$$\therefore\ V^2 = 2f \frac{h}{\sin a} = 2gh \frac{M+m}{M + m\sin^2 a}.$$

41. Since the particle is to be at rest relative to the plane it must have the same accelerations as the point of the plane with which it is in contact, *i.e.* $f\cos a$ along the plane upwards and $f\sin a$ perpendicular to the plane. Hence, if R be the reaction,

$$-R + mg\cos a = mf\sin a \text{ .... } (1),$$

and

$$\mu R - mg\sin a = mf\cos a \text{ .... } (2),$$

in the limiting case.

$$\therefore\ \mu(g\cos a - f\sin a) = g\sin a + f\cos a.$$

$$\therefore\ f = \frac{\mu g\cos a - g\sin a}{\mu\sin a + \cos a} \text{ .... } (3).$$

If f be greater than this value, then the ratio of the friction to the normal reaction, which by (1) and (2) $= \frac{g\sin a + f\cos a}{g\cos a - f\sin a}$, is $> \mu$, which is impossible, and so motion will ensue.

42. Let $ABCDEF$ be the hexagon, the side AB being on the ground. Let O be the point of projection and let it be nearer A than B. Let v and θ be the velocity and angle of projection, and let $OA = x$. Let $2a$ be the side of the hexagon, so that

$$AE = 2 \cdot 2a \sin 60° = 2a\sqrt{3}.$$

Draw FM perpendicular to OA, so that

$$FM = 2a \sin 60° = a\sqrt{3}, \quad \text{and} \quad AM = 2a \cos 60° = a,$$

and
$$\therefore\ OM = x - a.$$

Since the motion is symmetrical with respect to the vertical through the middle point of AB, the range is $2(x+a)$, and

$$\therefore\ \frac{2v^2 \sin\theta \cos\theta}{g} = 2(x+a) \quad \text{.....................} \quad (1).$$

Let t be the time from O to A, so that

$$x = v\cos\theta \cdot t, \quad \text{and} \quad v \sin\theta \cdot t - \frac{1}{2} g t^2 = 2a\sqrt{3}.$$

Therefore eliminating t, we have

$$x \tan\theta - \frac{1}{2} g \cdot \frac{x^2}{v^2 \cos^2\theta} = 2a\sqrt{3} \quad \text{.......... ...} \quad (2).$$

Similarly since the particle goes through F,

$$(x-a)\tan\theta - \frac{1}{2} g \frac{(x-a)^2}{v^2\cos^2\theta} = a\sqrt{3} \quad \text{................} \quad (3).$$

Subtracting (3) from (2), we have, after substituting for v^2 from (1) and simplifying,

$$x + a = a\frac{\sqrt{3}}{2}\tan\theta,$$

and therefore from (1)

$$v^2\cos^2\theta = g\frac{\sqrt{3}}{2}a.$$

Substituting for x and v in (2), we have

$$\left(\frac{\sqrt{3}}{2}\tan\theta - 1\right)\tan\theta - \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\tan\theta - 1\right]^2 = 2\sqrt{3}.$$

Simplifying, we have

$$\tan^2\theta = \frac{28}{3}, \quad \text{and} \quad \therefore\ \cos^2\theta = \frac{3}{31}.$$

$$\therefore\ \frac{\text{least velocity}}{\text{final velocity}} = \frac{v\cos\theta}{v} = \cos\theta = \frac{\sqrt{3}}{\sqrt{31}}.$$

43. Let m be the mass of the man, and therefore $\frac{3m}{2}$ the mass of the weight. Let T be the tension which he causes in the rope and f, f_1 the corresponding accelerations of the man and weight, both measured upwards.

Then $$T-mg=mf \quad \text{.....(1)},$$

and $$T-\frac{3m}{2}g=\frac{3m}{2}f_1 \quad \text{.....(2)}.$$

Also, since the acceleration of the point of the rope in contact with his hands must be the same as that of the weight,

$$\therefore f+f_1=\frac{6g}{7} \quad \text{.....(3)}.$$

Solving (1), (2), (3), we have $f_1=\frac{g}{7}$ and $T=\frac{12}{7}mg=\frac{12}{7}$ times the weight of the man.

44. Let u be the velocity of the sphere just before hitting the plane, v_1 the velocity just after hitting it, both being perpendicular to the inclined face. Let v_2 be the velocity communicated to the wedge by the impulse R between the wedge and sphere.

Then $$R=m(u+v_1) \quad \text{.....(1)},$$

and $$R\sin\alpha=Mv_2 \quad \text{.....(2)}.$$

Also, by Newton's experimental law,

$$v_1+v_2\sin\alpha=eu \quad \text{.....(3)}$$

Solving, we have

$$u : v_1 :: M+m\sin^2\alpha : eM-m\sin^2\alpha.$$

45. Let f_1 be the acceleration upwards of the 4 lbs. and therefore of the 1 lb. pulley downwards. Let f_2 and f_3 be the accelerations of the 2 and 3 lbs. both measured downwards. Let T, T_1 be the tensions of the upper and lower strings. Then

$$T-4g=4f_1 \quad \text{.....(1)},$$

$$2T_1-T+g=1\,.f_1 \quad \text{.....(2)},$$

$$2g-T_1=2f_2 \quad \text{.....(3)},$$

and $$3g-T_1=3f_3 \quad \text{.....(4)}.$$

Also, just as in Ex. 29, we have

$$f_2+f_3=2f_1 \quad \text{.....(5)}.$$

Adding (1), (2), (3), (4) we have

$$5f_1+2f_2+3f_3=2g \quad \text{.....(6)}.$$

Also from (3), (4),

$$3f_3-2f_2=g \quad \text{.....(7)}.$$

Solving (5), (6), (7) we have $f_1=\frac{9g}{49}$.

46. When the particle is distant x from the middle point the stretched lengths of the two parts of the string are $\frac{na}{2}+x$ and $\frac{na}{2}-x$ respectively, and the unstretched lengths are each $\frac{a}{2}$. Let T and T_1 be the corresponding tensions; then, by Hooke's Law,

$$T=\lambda\frac{\frac{na}{2}+x-\frac{a}{2}}{\frac{a}{2}},\ \text{and}\ T_1=\frac{\frac{na}{2}-x-\frac{a}{2}}{\frac{a}{2}}.$$

The acceleration of the particle then $=\frac{T-T_1}{m}=\frac{4\lambda}{ma}x$.

The motion is thus simple harmonic, and the time of oscillation

$$=2\pi\sqrt{\frac{ma}{4\lambda}}.$$

47. With the usual notation

$$[M'][L']^2[T']^{-2}=\frac{1}{2}.5\times 2g.50\,[M][L]^2[T]^{-2}\ \ldots\ldots\ (1).$$

$$[M'][L']\,[T']^{-1}=5\times\sqrt{2g.50}\,[M][L][T]^{-1}\ \ldots\ldots\ (2),$$

$$[L']\qquad=50\,[L]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Dividing (1) by (2), we have

$$[L'][T']^{-1}=\frac{1}{2}\sqrt{100g}\,[L][T]^{-1},$$

and then, by (3),

$$[T']^{-1}=\frac{\sqrt{100g}}{100}[T]^{-1}=\frac{40\sqrt{2}}{100}[T]^{-1}.$$

$$\therefore\ [T']=\frac{100}{40\sqrt{2}}[T]=\frac{5}{2\sqrt{2}}\text{ secs.}=\frac{5}{4}\sqrt{2}\text{ secs.}$$

48. Let C be the centre of the circle; draw CN perpendicular to OY.

Then the velocity of Y

$=$ the velocity of Y relative to N together with the velocity of N

$=$ the velocity of P relative to C together with velocity of N

$=$ the actual velocity of P together with the velocity of N.

Hence the velocity of Y relative to $P=$ the actual velocity of N.

Now N moves on a circle of half the size of the given circle; and the angular velocity of N in this circle

$=$ twice its angular velocity about O

$=$ twice the angular velocity of P about C, since ON, CP are always parallel.

Hence the velocity of N

$$=\frac{1}{2}\,OC\times\text{twice the angular velocity of } P$$

$$=CP\times\text{the angular velocity of } P=\text{the velocity of } P.$$

49. Let f and f' be the accelerations (both upwards) of the two men, and T the tension of the rope at any time.

Then $$T-Mg=Mf,$$

and $$T-(M+m)\,g=(M+m)\,f'.$$

By subtraction, we have

$$f'=\frac{M}{M+m}f-\frac{m}{M+m}g \quad \ldots(1)$$

Also $$h=\frac{1}{2}ft^2.$$

$$\therefore \text{ required distance}=h-\frac{1}{2}f't^2$$

$$=h-\frac{1}{2}\,\frac{M}{M+m}ft^2+\frac{1}{2}\,\frac{m}{M+m}gt^2$$

$$=h-\frac{M}{M+m}h+\frac{1}{2}\,\frac{m}{M+m}gt^2$$

$$=\frac{m}{M+m}\left[\frac{gt^2}{2}+h\right].$$

50. Let P be the pull of the engine in poundals, and f the retardation per unit mass due to friction. Since the train was originally travelling with uniform velocity, the forces on it then just balanced, and therefore $P=Mf$. After the uncoupling, the distance gone by the last carriage $=\frac{v^2}{2f}$, where v was the original velocity.

The acceleration of the first portion of the train

$$=\frac{P-(M-m)f}{M-m}=\frac{m}{M-m}f=f'.$$

Hence the velocity v_1 after a distance l has been described is given by

$$v_1^2=v^2+2f'l.$$

Hence the distance described by the first part before coming to rest

$$=l+\frac{v_1^2}{2f}=l+\frac{v^2}{2f}+l\frac{f'}{f}$$

$$=l+\frac{v^2}{2f}+\frac{ml}{M-m}=\frac{v^2}{2f}+\frac{Ml}{M-m}. \quad \therefore \text{ etc.}$$

51. Let T be the tension of the string; f_1 and f_2 the accelerations of m and m', and f_3 that of the pulley.

Then
$$mg - T = mf_1 \qquad \ldots\ldots (1),$$
$$m'g - T = m'f_2 \qquad \ldots\ldots (2),$$
and
$$2T = Mf_3 \qquad \ldots\ldots (3).$$

Also, as in Ex. 29, since the length of the string is constant,
$$2f_3 = f_1 + f_2 \qquad \ldots\ldots (4).$$

From (1), (2), and (3), substituting in (4), we have
$$\frac{4T}{M} = g - \frac{T}{m} + g - \frac{T}{m'}.$$
$$\therefore\ T = \frac{2Mmm'}{4mm' + Mm + Mm'}g.$$
$$\therefore\ f_3 = \frac{2T}{M} = \frac{4mm'}{M(m+m') + 4mm'}g.$$

52. The resultant acceleration is $g' = \sqrt{f^2 + g^2}$ at an angle θ to the horizon, such that $\tan\theta = \frac{g}{f}$.

The velocity resolved perpendicular to the direction of g'

$= v\cos[90° - \alpha - \theta] = v\sin(\alpha + \theta)$

$= v[\sin\alpha + \cos\alpha\tan\theta]\,.\cos\theta = v\,\dfrac{f\sin\alpha + g\cos\alpha}{f}\,\dfrac{f}{\sqrt{f^2+g^2}}$

$= v\,\dfrac{f\sin\alpha + g\cos\alpha}{g'}$.

Hence, as in Art. 113, the latus rectum
$$= \frac{2}{g'}\left[v\,\frac{f\sin\alpha + g\cos\alpha}{g'}\right]^2 = \frac{2v^2(f\sin\alpha + g\cos\alpha)^2}{(f^2+g^2)^{\frac{3}{2}}}.$$

53. The acceleration of the particle perpendicular to the inclined face

$$= \text{that of the wedge in this direction} = f\sin\alpha.$$

The acceleration of the particle down the plane $= g\sin\alpha$.
Hence the total vertical acceleration of the particle
$$= f\sin\alpha\cos\alpha - g\sin^2\alpha = \sin\alpha\cos\alpha\,[f - g\tan\alpha].$$

Hence the particle ascends vertically if $f > g\tan\alpha$.

Let O be the lowest point of the inclined face OA, so that $OA = \dfrac{h}{\sin\alpha}$, and let P be the point at which the particle is at the end of time t.

The acceleration of the particle up the inclined face relative to the plane $= f\cos\alpha - g\sin\alpha$ Hence

$$OP = \frac{1}{2}(f\cos\alpha - g\sin\alpha)\,t^2.$$

The velocity relative to the plane at time $t = (f\cos\alpha - g\sin\alpha)\,t$ After the end of time t the acceleration relative to the plane

$$= -g\sin\alpha$$

The particle just reaches the top if

$$(f\cos\alpha - g\sin\alpha)^2\,t^2 = 2\left[\frac{h}{\sin\alpha} - OP\right]g\sin\alpha$$

$$= 2gh - g\sin\alpha\,(f\cos\alpha - g\sin\alpha)\,t^2,$$

i e if etc

54. Let a and b ft be the radii of the wheel and of the axle respectively, so that

$$a \quad 2 = b \quad 10 \qquad (1)$$

When the 1 lb has been added let T, T' be the tensions of the strings going round the wheel and axle, and f and f_1 the accelerations of the 3 and 10 lbs respectively downwards and upwards

Then $$3g - T = 3f \qquad (2)$$

and $$T_1 - 10g = 10f_1 \qquad (3)$$

The tensions must balance about the centre of the machine, so that

$$T \quad a = T \quad b,$$

i e $$T' = 5T, \text{ by } (1) \qquad (4)$$

Also since the whole machine turns about the centre,

$$\therefore \frac{f}{f_1} = \frac{a}{b} = 5 \qquad (5)$$

(2), (3), and (4) give $5g = 15f + 10f_1 = 15f + 2f$ by (5).

$$\therefore f = \frac{5g}{17}$$

Then, by (2), $$T = 3g - \frac{15g}{17} = 2\tfrac{2}{17} \text{ lbs wt.},$$

and $$T_1 = 10\tfrac{10}{17} \text{ lbs. wt.}$$

55. If W be the mass supported, then as in *Statics*, Art. 164,

$$W(a-b)=2Pc \qquad (1).$$

When P becomes $2P$ let T be the tension of its rope, and T' that of the other. Let f be the acceleration of $2P$ downwards and f' that of W upwards. Then

$$2Pg-T=2Pf \qquad (2),$$

and

$$2T'-W.g=W.f' \qquad (3).$$

Also by the Principle of Virtual Work [*Statics*, Art. 236], we have

$$\frac{f}{f'}=\frac{W}{P}=\frac{2c}{a-b} \qquad (4).$$

Also, since the moments of the tensions of the strings round the central line of the axis must vanish,

$$\therefore T.c=T'(a-b) \qquad (5).$$

Solving (2), (3), (4), (5) for f, and using the relation (1) for $\frac{P}{W}$ we have the required answer.

56. Let O be the point of projection, OA the range on the inclined plane; let v be the velocity of projection at an angle β to the inclined plane. The time of flight is $2\frac{v\sin\beta}{g\cos\alpha}$, and therefore velocity at A along the plane

$$=v\cos\beta-\frac{2v\sin\beta}{g\cos\alpha}.g\sin\alpha.$$

Hence the condition gives

$$v\sin\beta : v\cos\beta-\frac{2v\sin\beta\sin\alpha}{\cos\alpha} :: \cot\alpha : 1.$$

$$\therefore \cos\alpha\cos\beta=3\sin\alpha\sin\beta.$$

$$\therefore \frac{\sin\beta}{\cos\alpha}=\frac{\cos\beta}{3\sin\alpha}=\frac{1}{\sqrt{1+8\sin^2\alpha}}.$$

After hitting the plane again at A the velocity along the plane downwards initially

$$=v\cos\beta-\frac{2v\sin\beta\sin\alpha}{\cos\alpha}, \text{ as before,}$$

$$=\frac{v\sin\alpha}{\sqrt{1+8\sin^2\alpha}}.$$

Also the initial velocity perpendicular to the plane $=v\sin\beta$, as before, and hence time of flight in second trajectory $=\frac{2v\sin\beta}{g\cos\alpha}$.

Therefore second range

$$=\frac{v\sin\alpha}{\sqrt{1+8\sin^2\alpha}}\times\frac{2v\sin\beta}{g\cos\alpha}+\frac{1}{2}g\sin\alpha\,.\left(\frac{2v\sin\beta}{g\cos\alpha}\right)^2$$

$$=\frac{2v^2}{g}\tan\alpha\,.\,\frac{\cos\alpha}{1+8\sin^2\alpha}+\frac{2v^2}{g\cos^2\alpha}\,.\,\frac{\cos^2\alpha\,.\,\sin\alpha}{1+8\sin^2\alpha}$$

$$=\frac{4v^2}{g}\,\frac{\sin\alpha}{1+8\sin^2\alpha}.$$

But the range OA in the original trajectory, by Art. 109,

$$=\frac{2v^2}{g}\,\frac{\cos(\alpha+\beta)\,.\,\sin\beta}{\cos^2\alpha}=\frac{4v^2}{g}\,\frac{\sin\alpha}{1+8\sin^2\alpha}.$$

Also the time of flight in either trajectory, and also in the vertical path after leaving A,

$$=\frac{2v\sin\beta}{g\cos\alpha}=\frac{2v}{g\sqrt{1+8\sin^2\alpha}}.$$

$\therefore$ total time = three times this = etc.

57. Let f_1, f_2, f_3, f_4 be the accelerations, all measured downwards, of the masses $2m$, $3m$, m, and $4m$; f_5 and f_6 those of the two pulleys; T the tension of the string round the fixed pulley, T_1 of that round the first movable pulley, and T_2 of that round the second one.

Then
$$2mg-T_1=2mf_1 \qquad \text{(1)},$$
$$3mg-T_1=3mf_2 \qquad \text{(2)},$$
$$2T_1-T+mg=mf_5 \qquad \text{(3)},$$
$$mg-T_2=mf_3 \qquad \text{(4)},$$
$$4mg-T_2=4mf_4 \qquad \text{(5)},$$
and
$$2T_2-T+mg=mf_6 \qquad \text{(6)}.$$

Also, just as in Ex. 29, we have, since the strings are of constant length,
$$f_1+f_2=2f_5 \qquad \text{(7)},$$
and
$$f_3+f_4=2f_6 \qquad \text{(8)}.$$
Also
$$f_5=-f_6 \qquad \text{(9)},$$
since one of the movable pulleys goes up as fast as the other goes down.

Solving these equations, we have $f_5=\frac{4g}{25}$.

58. Let O be the centre of the circle, OA the vertical; if possible let the bead be on the point of slipping when at P where $\angle AOP=\theta$. Let R be the reaction along OP, and μR the friction. Then the acceleration of the bead towards the centre is $\omega^2 a$, and along the tangent it is zero.

$$\therefore\ m\omega^2 a = mg\cos\theta - R \quad\ldots\ldots\ldots\ldots(1),$$

and

$$\mu R = mg\sin\theta \quad\ldots\ldots\ldots\ldots(2),$$

$$\therefore\ \omega^2 a = \frac{g}{\mu}(\mu\cos\theta-\sin\theta) = \frac{g\sqrt{1+\mu^2}}{\mu}\cos(\theta+\epsilon),\ \text{where } \mu=\cot\epsilon.$$

Hence the position of P is given by $\cos(\theta+\epsilon)=\dfrac{\mu\omega^2 a}{g\sqrt{1+\mu^2}}$.

This gives no real value of θ, *i.e.* the bead is never on the point of slipping,

if
$$\mu\omega^2 a > g\sqrt{1+\mu^2},$$

i.e. if
$$\omega^2 > \frac{g}{a}\sqrt{1+\frac{1}{\mu^2}},$$

i.e. if
$$\omega > \sqrt{\frac{g}{a}}\sqrt[4]{1+\frac{1}{\mu^2}}.$$

59. Let O be the centre of the hoop, AOB a vertical diameter, A being the lowest point. Let the particle leave the hoop at P.

If the velocity of projection at A be u, the velocity v at P is given by $v^2=u^2-2g(a+a\cos\theta)$, where $\angle BOP=\theta$.

The particle leaves the hoop at P, if

$$mg\cos\theta=\frac{mv^2}{a}=m\frac{u^2}{a}-2g(1+\cos\theta),$$

i.e. if
$$u^2=ga[2+3\cos\theta] \quad\ldots\ldots\ldots\ldots(1).$$

Let t be the time from P to A; then at P the particle starts with initial horizontal and vertical velocities equal to $v\cos\theta$ and $v\sin\theta$, and describes a parabolic path. Thus

$$a\sin\theta=v\cos\theta\,.\,t,\ \text{and}\ -(a+a\cos\theta)=v\sin\theta\,.\,t-\frac{1}{2}gt^2.$$

$$\therefore\ -(a+a\cos\theta)=\frac{a\sin^2\theta}{\cos\theta}-\frac{1}{2}g\frac{a^2\sin^2\theta}{v^2\cos^2\theta}\quad\ldots\ldots\ldots\ldots(2).$$

But from (1) $v^2=ag\cos\theta$.

$$\therefore\ -a-a\cos\theta=\frac{a}{\cos\theta}-a\cos\theta-\frac{1}{2}\frac{ga^2\sin^2\theta}{ag\cos^3\theta}.$$

$$\therefore\ -2a\cos^3\theta=2a\cos^2\theta-a\sin^2\theta.$$

$$\therefore\ 2\cos^3\theta+3\cos^2\theta-1=0.$$

$$\therefore\ (\cos\theta+1)^2(2\cos\theta-1)=0.$$

The only admissible solution is $\cos\theta=\frac{1}{2}$, *i.e.* $\theta=60°$, and then,

by (1),
$$u^2=\frac{7ga}{2}.$$

60. With the notation of the last example we have $u^2=2g(a+b)$, and the equation (1) becomes

$$\cos\theta=\frac{2b}{3a} \quad \ldots\ldots \qquad (1),$$

and

$$v^2=ag\cos\theta=\frac{2}{3}bg \quad \ldots\ldots \qquad (2).$$

The string will be taut again at the point of intersection Q of the circle and parabolic path. Hence if $\angle AOQ=\phi$, and t be the time from P to Q, then

$$a\sin\theta+a\sin\phi=v\cos\theta\,.\,t,$$

and

$$-(a\cos\theta+a\cos\phi)=v\sin\theta\,.\,t-\frac{1}{2}gt^2.$$

Eliminating ϕ we have

$$(v\cos\theta\,.\,t-a\sin\theta)^2+\left(a\cos\theta+v\sin\theta\,.\,t-\frac{1}{2}gt^2\right)^2=a^2.$$

$$\therefore\ v^2t^2+a^2-gt^2(a\cos\theta+v\sin\theta\,.\,t)+\frac{1}{4}g^2t^4=a^2,$$

i.e. $\quad \frac{1}{4}g^2t^2=-v^2+g(a\cos\theta+v\sin\theta\,.\,t)$

$$=gv\sin\theta\,.\,t=g\,.\,\sqrt{\frac{2bg}{3}}\sqrt{1-\frac{4b^2}{9a^2}}\,.\,t.$$

$$\therefore\ 27gt^2a^2=32bg(9a^2-4b^2).$$

61. Let O be the point of attachment. Draw $OABC$ vertical, and let $OA=a$, $AB=a$, $BC=2a$.

Let P be any point between B and C such that $AP=x$. Let T be the tension at P. Then the acceleration of P towards O

$$=\frac{T-mg}{m}=\frac{\lambda\frac{x}{a}-mg}{m}, \text{ by Hooke's Law,}$$

$$=g\frac{x-a}{a}=\frac{g}{a}\,.\,BP, \text{ since } \lambda=mg.$$

The motion is therefore simple harmonic about B as centre with $2a$ as half-amplitude. Hence, by page 196, the time from C to A

$$=\frac{1}{\sqrt{\frac{g}{a}}}\cos^{-1}\left(\frac{-a}{2a}\right)=\sqrt{\frac{a}{g}}\times\frac{2\pi}{3}.$$

Also the velocity at $A=\sqrt{\frac{g}{a}}\sqrt{4a^2-a^2}=\sqrt{3ga}.$

After passing A the string becomes slack and this velocity is destroyed in time $\sqrt{3ga} \div g = \sqrt{\frac{3a}{g}}$. Therefore total time from C to the highest point of the path $= \sqrt{\frac{a}{g}}\left[\sqrt{3} + \frac{2\pi}{3}\right]$.

Also required time = twice this. $\therefore$ etc.

62. Let v be the velocity with which the ball starts, so that $T = \frac{mv}{10}$ is the impulse on the system A and B. The velocity of each of the latter is therefore $\frac{v}{20}$, and they move with no acceleration.

Let x be the distance A goes before the ball catches him. Then, since the ball *just* reaches him, the time that elapses is that in which v is reduced to $\frac{v}{20}$ by the acceleration g and therefore is $\frac{19v}{20g}$.

Hence $$v^2 - \left(\frac{v}{20}\right)^2 = 2g(x+h) \quad \ldots\ldots (1),$$

and $$x = \frac{v}{20} \times \frac{19v}{20g} \quad \ldots\ldots (2).$$

$$\therefore\ 2g(x+h) = \frac{399}{400}v^2 = \frac{399 \times gx}{19} = 21gx.$$

$$\therefore\ x = \frac{2h}{19}.$$

After A has caught the ball his acceleration downwards

$$= \frac{m + \frac{m}{10} - m}{m + \frac{m}{10} + m}\, g = \frac{g}{21},$$

and the velocity $\frac{v}{20}$ is therefore destroyed in a space y, where

$$\left(\frac{v}{20}\right)^2 = 2 \cdot \frac{g}{21} \cdot y,$$

and $$\therefore\ y = \frac{21}{2g} \cdot \frac{v^2}{400} = \frac{21}{2g} \cdot \frac{gx}{19} = \frac{21}{38}x = \frac{21h}{361}.$$

$$\therefore \text{ required total distance} = x + y = \frac{59h}{361}.$$

63. Let f_1 and f_2 be the accelerations, both measured downwards, of m and m'. Since the mass M is to remain at rest, the tension of the string must be $\frac{M}{2}g$.

Hence
$$mg-\frac{Mg}{2}=mf_1 \quad\ldots\ldots (1),$$
and
$$m'g-\frac{Mg}{2}=m'f_2 \quad\ldots\ldots (2).$$

Also, since one of the moving masses goes up as fast as the other goes down,
$$\therefore f_1=-f_2,$$
i.e.
$$f_1+f_2=0 \quad\ldots\ldots (3).$$

Hence, from (1) and (2),
$$g-\frac{Mg}{2m}+g-\frac{Mg}{2m'}=0.$$
$$\therefore \frac{4}{M}=\frac{1}{m}+\frac{1}{m'}.$$

64. Let f be the acceleration of the wedge horizontally and therefore that of M' vertically. Let T be the tension of the string, R the reaction between the wedge and the mass m, and f_1 and f_2 the accelerations of the mass m perpendicular to and down the inclined face of the wedge.

Then
$$R-mg\cos\alpha=mf_1 \quad\ldots\ldots (1),$$
$$g\sin\alpha=f_2 \quad\ldots\ldots (2),$$
$$R\sin\alpha+T=Mf \quad\ldots\ldots (3),$$
and
$$M'g-T=M'f \quad\ldots\ldots (4).$$

Also, since the acceleration f_1 must be equal to the acceleration of the wedge perpendicular to the inclined plane,
$$\therefore f_1=-f\sin\alpha \quad\ldots\ldots (5).$$

Solving (1), (3), (4), (5), we have
$$f=\frac{M'g+mg\cos\alpha\sin\alpha}{M+M'+m\sin^2\alpha}.$$

Now the relative acceleration

$=$acceleration of the wedge resolved upwards along the inclined face $+f_2$

$$=f\cos\alpha+g\sin\alpha$$
$$=\frac{(M+M'+m)\sin\alpha+M'\cos\alpha}{M+M'+m\sin^2\alpha}g.$$

Also from (1)
$$R=mg\cos\alpha+mf_1=m(g\cos\alpha-f\sin\alpha)$$
$$=m\frac{(M+M')\cos\alpha-M'\sin\alpha}{M+M'+m\sin^2\alpha}g.$$

65. With the notation of Ex. 31, page 97, the acceleration of the particle relative to the plane is down the line of greatest slope and

$$=g\sin\alpha+f_2\cos\alpha$$

$$=g\sin\alpha+\frac{mg\sin\alpha\cos^2\alpha}{M+m\sin^2\alpha}$$

$$=g\sin\alpha\times\frac{M+m}{M+m\sin^2\alpha},$$

and is therefore constant and in a constant direction. Hence, as in Art. 112, the path on the plane is a parabola.

66. The particle cannot retrace its path unless its direction of motion at the point of impact is perpendicular to the plane.

Let v be the initial velocity at an angle β to the plane. The time of flight is $\frac{2v\sin\beta}{g\cos\alpha}$ and hence, as in Art. 109,

$$l\cos\alpha=\frac{2v^2\sin\beta\cos(\alpha+\beta)}{g\cos\alpha}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Also at the moment of impact the velocity along the plane

$$=v\cos\beta-g\sin\alpha\,.\,\frac{2v\sin\beta}{g\cos\alpha}.$$

This is zero if

$$2\sin\alpha\sin\beta=\cos\alpha\cos\beta,$$

i.e. if

$$\frac{\sin\beta}{\cos\alpha}=\frac{\cos\beta}{2\sin\alpha}=\frac{1}{\sqrt{1+3\sin^2\alpha}},$$

and then from (1)

$$v^2=\frac{gl}{2}\,\frac{\cos^2\alpha}{\sin\beta(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}$$

$$=\frac{gl}{2}\,\frac{1+3\sin^2\alpha}{\sin\alpha}.$$

67. Let λ be the coefficient of elasticity so that

$$mg=\lambda\frac{e}{a}\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

When the mass is at a distance x from A the point where the perpendicular from O, the fixed point, upon the bar meets it, the length of the spring is $\sqrt{c^2+x^2}$ and its tension

$$=\lambda\frac{\sqrt{c^2+x^2}-a}{a}=\frac{mg}{e}[\sqrt{c^2+x^2}-a].$$

Hence the force to bring the body back

$$=T\cos OMA=T\frac{x}{\sqrt{c^2+x^2}}=\frac{mg}{e}\left[x-\frac{ax}{\sqrt{c^2+x^2}}\right]=\frac{mg}{e}\left[x-\frac{ax}{c}\right],$$

as far as first powers of x.

Hence the acceleration $=\frac{m}{M}\frac{g}{e}\frac{c-a}{c}\times x$

The motion is thus simple harmonic for small values of x, and the time is as given

68. Let P be the difference between the horizontal pressures exerted by the two rails Then P is equal to a parallel force P acting through the centre of inertia together with a couple P h tending to overturn the carriage

Also $$P=m\frac{v^2}{r} \tag{1}$$

Let R and S be the vertical pressures exerted by the inner and outer rails, so that $R+S=mg$

The forces R, S, and mg must therefore form a couple to balance the former couple

Taking moments about the outer rail the moment of this couple

$$=mga-R\ 2a$$

$$m\frac{v^2}{r}h=mga-2Ra,$$

$$R=\frac{m}{2}\left[g-\frac{v^2h}{ra}\right].$$

If $v>\sqrt{\frac{gra}{h}}$, this value of R will be negative, i e for equilibrium the inner rail would have to pull the carriage downwards, which is impossible Hence the carriage would upset

69. Suppose the wedge to move with acceleration f Let S be the pressure of the table on the wedge and R the reaction between the inclined face and particle, also let f_1 be the acceleration of the particle perpendicular to the inclined face

Then $$R-mg\cos\alpha=mf_1 \tag{1},$$

$$Mg+R\cos\alpha=S \tag{2},$$

and $$R\sin\alpha-S\tan\epsilon=Mf \tag{3}$$

Also since the wedge and mass remain in contact, the acceleration of each perpendicular to the plane face must be the same

Hence $$f_1=-f\sin\alpha. \tag{4}$$

From (2) and (3),

$$R[\cos\alpha\sin\epsilon - \sin\alpha\cos\epsilon] = -Mf\cos\epsilon - Mg\sin\epsilon,$$

and hence, by (1) and (4),

$$(mg\cos\alpha - mf\sin\alpha)\sin(\alpha-\epsilon) = Mf\cos\epsilon + Mg\sin\epsilon.$$

$$\therefore\ f = \frac{mg\cos\alpha\sin(\alpha-\epsilon) - Mg\sin\epsilon}{M\cos\epsilon + m\sin\alpha\sin(\alpha-\epsilon)}.$$

The wedge will just not move if this acceleration be just zero, *i.e.* if

$$m\cos\alpha\sin(\alpha-\epsilon) = M\sin\epsilon,$$

i.e. if $$m\cos\alpha[\sin\alpha - \cos\alpha\tan\epsilon] = M\tan\epsilon,$$

i.e. if $$\tan\epsilon = \frac{m\cos\alpha\sin\alpha}{M + m\cos^2\alpha}.$$

If the wedge is to move, the coefficient of friction must be less than this.

70. Let T be the tension of the unbroken string which is attached to the point O of the window $OABC$, where $OA\ (=a)$ is vertical and $OC\ (=b)$ horizontal. The window will be in contact with the framework at A and C. Let R and S be the normal reactions at these points and μR, μS the frictions both of which act upwards. Let T be the tension of the unbroken string; then f is also the upward acceleration of the weight $\frac{M}{2}$, where M is the mass of the window.

Then $$T - \frac{M}{2}g = \frac{M}{2}f \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$Mg - T - \mu(R+S) = Mf \quad \ldots\ldots\ldots\ldots (2).$$

Also, by resolving horizontally,

$$R = S \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also, since the frame cannot twist, the sum of the moments about the centre of gravity must vanish.

$$\therefore\ (T+\mu R)\frac{b}{2} = \mu S\frac{b}{2} + (R+S)\frac{a}{2}.$$

$$\therefore\ T.b = 2aR, \text{ by (3)}, \quad \ldots\ldots\ldots\ldots (4).$$

(1), (2) and (3) give

$$\frac{Mg}{2} - 2\mu R = \frac{3M}{2}f.$$

This and T from (1) substituted in (4) give

$$b\frac{M}{2}(g+f) = 2\times\frac{a}{2\mu}\frac{M}{2}(g-3f).$$

$$\therefore\ \mu = \frac{a}{b}\cdot\frac{g-3f}{g+f}.$$

71. The average forces whilst the body is describing the 1st, 2nd, 3rd etc., feet are

$$\frac{450+320}{2}, \quad \frac{320+270}{2}, \ldots \text{ lbs. wt.}$$

i e. 385, 295, 340, 445, 545, 755 lbs. wt.

Hence the net vertical forces are

85, -5, 40, 145, 245, and 455 lbs. wt.

When at a height of $5\frac{1}{2}$ feet the potential energy stored up

$$=5\tfrac{1}{2} \times 300 \text{ ft.-lbs.} = 1650 \text{ ft.-lbs.}$$

The kinetic energy of the mass then = work done on it

$$=85 \times 1 + (-5) \times 1 + 40 \times 1 + 145 \times 1 + 245 \times 1 + 455 \times \tfrac{1}{2} \text{ ft.-lbs.}$$

$$=737{\cdot}5 \text{ ft.-lbs.}$$

The total work done by the force = sum of the kinetic and potential energies

$$=2387{\cdot}5 \text{ ft.-lbs.}$$

72. The average forces during consecutive inches are

$$\frac{22+36{\cdot}2}{2}, \quad \frac{36{\cdot}2+44{\cdot}5}{2}, \ldots$$

i e. 29·1, 40·35, 46·75, 50·5, 51·9 and 49·9 lbs. wt.

At a height of 2 inches the work done by the force

$$=\frac{29{\cdot}1+40{\cdot}35}{12}=\frac{69{\cdot}45}{12} \text{ ft.-lbs.}$$

The work done against the weight $=\dfrac{10 \times 2}{12}=\dfrac{20}{12}$ ft -lbs.

The work done against the spring (*Statics*, Art. 217)

$$=\frac{1}{2}[20+0] \times \frac{2}{12}=\frac{20}{12} \text{ ft.-lbs.}$$

$\therefore$ Kinetic energy = net work done

$$=\frac{69{\cdot}45-20-20}{12}=2{\cdot}45 \ldots \text{ ft.-lbs.}$$

The potential energy then

$$=\frac{20+20}{12}=3\tfrac{1}{3} \text{ ft.-lbs.}$$

At a height of 4 inches the corresponding works are

$$\frac{29\cdot1+40\cdot35+46\cdot75+50\cdot5}{12},\ \frac{10\times4}{12},\ \frac{1}{2}[40+0]\times\frac{4}{12};$$

i.e.

$$\frac{166\cdot7}{12},\ \frac{40}{12},\ \frac{80}{12}.$$

$\therefore$ Kinetic energy = net work done

$$=\frac{166\cdot7-40-80}{12}=\frac{46\cdot7}{12}=3\cdot89\ \ldots\ \text{ft.-lbs.}$$

The Potential Energy then $=\dfrac{40+80}{12}=10$ ft.-lbs.

Also, if V be the velocity when it has been raised 6 inches, we have

$$\frac{1}{2}\cdot10V^2=\frac{166\cdot7+51\cdot9+49\cdot9}{12}-\frac{10\times6}{12}-\frac{1}{2}[60+0]\frac{6}{12}$$

$$=\frac{268\cdot5-60-180}{12}=\frac{28\cdot5}{12}\text{ ft.-lbs.}$$

$$=\frac{28\cdot5}{12}\times32\text{ ft. poundals.}$$

$$\therefore\ V^2=\frac{5\cdot7\times32}{12}=1\cdot9\times8=15\cdot2.$$

$$\therefore\ V=\sqrt{15\cdot2}=3\cdot9\text{ ft. per sec. nearly.}$$

73. If A be the area of the section of the hose, and m the mass of a unit volume of water, the water delivered per second $=Amv$, each particle of which has velocity v.

$$\therefore\ \text{energy of this water}=\frac{1}{2}.Amv.v^2=\frac{Am}{2}.v^3,$$

so that the energy delivered per second varies as v^3.

74. The force between the wheels and the rails

$$=4\text{ tons wt.}=4\times2240\text{ lbs. wt.}$$

Hence if v be the velocity of the train in ft. per sec., we have

$$4\times2240\times v=\text{work done by the train per second}=700\times550.$$

$$\therefore\ v=\frac{700\times550}{4\times2240}=\text{very nearly 43 ft. per second}$$

$$=\text{a little less than 30 miles per hour.}$$

75. Let the engine exert its pull for time t_1, during which it goes a distance x, and a velocity v is acquired. Then the accelerations are $\dfrac{P-R}{M}g$ and $\dfrac{-R}{M}g$,

so that
$$\frac{(P-R)g}{M}t_1 = v = \frac{R}{M}g(t-t_1) \qquad (1),$$

and
$$\frac{(P-R)g}{M}.x = \frac{1}{2}v^2 = \frac{Rg}{M}(l-x) \qquad .. \quad (2).$$

$$\left.\begin{aligned} \therefore\ v\left[\frac{M}{P-R}+\frac{M}{R}\right] &= gt, \\ \frac{1}{2}v^2\left[\frac{M}{P-R}+\frac{M}{R}\right] &= gl \end{aligned}\right\}$$
and

Hence $\dfrac{g^2t^2}{gl} = 2\left[\dfrac{M}{P-R}+\dfrac{M}{R}\right]$, whence $P = \dfrac{R^2gt^2}{Rgt^2-2Ml}$.

Also (1) gives
$$\frac{Pg}{M}t_1 = \frac{R}{M}gt, \text{ so that } t_1 = t\frac{R}{P} = t\frac{R^2gt^2-2MRl}{R^2gt^2} = t - \frac{2Ml}{Rgt}.$$

76. Whilst he is running down the hill uniformly the forces must balance, so that the resistance R of the air $= \frac{1}{m}Mg$. When he is going up the incline the total resistance

$$= R + \frac{1}{n}.Mg = \left(\frac{1}{m}+\frac{1}{n}\right)Mg.$$

$$\therefore \text{ work he does per second} = \left(\frac{1}{m}+\frac{1}{n}\right)Mg \times v.$$

$$\therefore \text{ required H.P.} = \left(\frac{1}{m}+\frac{1}{n}\right)\frac{Mgv}{550}.$$

77. Let H be his constant horse-power, and P the resistance. Then

$$\frac{88}{5}\times P = H = \frac{88}{12}\times\left(\frac{180}{40}+P\right).$$

$$\therefore\ 12P = 5P + 22\tfrac{1}{2}, \text{ so that } P = \frac{22\frac{1}{2}}{7} = 3\tfrac{3}{14}.$$

Also, when he is going down the second incline, the resistance to his motion

$$= P - \frac{1}{100}\ 180 = 3\tfrac{3}{14} - \frac{9}{5} = 1\tfrac{29}{70} \text{ lbs. wt.}$$

Hence if v be his required greatest velocity in ft. per sec., we have

$$1\tfrac{29}{70}\times v = H = \frac{88}{5}\times P = \frac{88}{5}\times 3\tfrac{3}{14}.$$

$$\therefore\ v=\frac{70}{99}\times\frac{88}{5}\times\frac{45}{14}\times\frac{60\times 60}{5280}\text{ miles per hour.}$$

$$=\frac{300}{11}=27\tfrac{3}{11}\text{ miles per hour.}$$

78. Let V be the velocity acquired. Since no momentum is lost by the impact,

$$\therefore\ mv=(M+m)\,V,\text{ so that }V=\frac{mv}{M+m}.$$

Let F be the resistance, so that $F.a=$ work done by it $=$ change in the kinetic energy of the system

$$=\frac{1}{2}mv^2-\frac{1}{2}(M+m)\,V^2$$

$$=\frac{1}{2}mv^2\left[1-\frac{m}{M+m}\right]=\frac{1}{2}\frac{Mm}{M+m}v^2.$$

Hence
$$F=\frac{Mm}{M+m}\frac{v^2}{2a}\text{ poundals}$$

$$=\frac{Mm}{M+m}\frac{v^2}{2ga}\text{ lbs. wt.}$$

Also, if t be the time of penetration,

$$F.t=\text{change in the momentum of }m$$

$$=m(v-V)$$

$$=\frac{mv}{M+m}[M+m-m]=\frac{Mmv}{M+m}.$$

$$\therefore\ t=\frac{Mmv}{M+m}\div\left(\frac{Mm}{M+m}\frac{v^2}{2a}\right)=\frac{2a}{v}.$$

During this time the acceleration of M

$$=\frac{F}{M}=\frac{m}{M+m}\frac{v^2}{2a}.$$

$$\therefore\ \text{distance described by it}=\frac{1}{2}\cdot\frac{m}{M+m}\frac{v^2}{2a}t^2$$

$$=\frac{m}{M+m}\frac{v^2}{4a}\cdot\frac{4a^2}{v^2}=\frac{ma}{M+m}\text{ feet.}$$

CAMBRIDGE: PRINTED BY W. LEWIS, M.A., AT THE UNIVERSITY PRESS